SOLUTIONS

No problem is ever permanently closed. Any comments, new solutions, or new insights on old problems are always welcomed by the problem editor.

53. [1993, 39] *Proposed by Russell Euler, Northwest Missouri State University, Maryville, Missouri.*

Prove analytically that

$$\sqrt[3]{19+9\sqrt{6}}+\sqrt[3]{19-9\sqrt{6}}$$

is an integer.

Solution I by Bob Prielipp, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin; Gregory Bruton, Cape Girardeau, Missouri; and Sherri Palmer, Ste. Genevieve, Missouri.

Since $(1+\sqrt{6})^3 = 19+9\sqrt{6}$ and $(1-\sqrt{6})^3 = 19-9\sqrt{6}$, the desired sum equals 2.

Solution II by Robert L. Doucette, McNeese State University, Lake Charles, Louisiana; Donald P. Skow, University of Texas-Pan American, Edinburg, Texas; Kanadasamy Muthuvel, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin; Joseph E. Chance, University of Texas-Pan American, Edinburg, Texas; Seung-Jin Bang, Albany, California; Joe Howard, New Mexico Highlands University, Las Vegas, New Mexico; J. Sriskandarajah, University of Wisconsin Center-Richland, Richland Center, Wisconsin; and the proposer.

Let $s = \sqrt[3]{19+9\sqrt{6}}$ and $t = \sqrt[3]{19-9\sqrt{6}}$. Note that $st = \sqrt[3]{19^2-9^2 6} = -5$ and $s^3+t^3 = (19+9\sqrt{6})+(19-9\sqrt{6}) = 38$. Since

$$(s+t)^3 = s^3+3s^2t+3st^2+t^3 = (s^3+t^3)+3st(s+t) = 38-15(s+t),$$

we see that $s+t$ is a root of the equation $x^3+15x-38=0$. Noting that $x=2$ is a root of this equation, we have $x^3+15x-38 = (x-2)(x^2+2x+19)$. Since $x^2+2x+19$ has no real roots, it follows that $s+t=2$.