ON THE EVALUATION OF MULTIPLE INTEGRALS

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1. Introduction. The students in my Mathematical Statistics course came across the following double integrals, and were unable to evaluate them using techniques that they had learned in calculus courses. The two integrals were

$$\int_0^1 \int_0^1 \frac{\theta^2 (x_1 x_2)^{\theta-1}}{\ln(x_1 x_2)} dx_1\, dx_2 \quad \text{and} \quad \int_0^\infty \int_0^\infty \frac{\theta^2 [(1+x_1)(1+x_2)]^{-(\theta+1)}}{\ln[(1+x_1)(1+x_2)]} dx_1\, dx_2$$

where $\theta > 0$. The computer algebra systems Derive and Mathematica were unable to evaluate the integrals. However, using results from Probability Theory the two integrals can be evaluated. We will generalize the two double integrals and instead consider the two multiple integrals

$$\int_0^1 \cdots \int_0^1 \frac{\theta^n (x_1 \cdots x_n)^{\theta-1}}{\ln(x_1 \cdots x_n)} dx_1 \cdots dx_n$$

and

$$\int_0^\infty \cdots \int_0^\infty \frac{\theta^n [(1+x_1)\cdots(1+x_n)]^{-(\theta+1)}}{\ln[(1+x_1)\cdots(1+x_n)]} dx_1 \cdots dx_n$$

where $\theta > 0$ and n is an integer greater than or equal to 2.

The key probability result needed to evaluate the integrals is the following.

Let $X_1, \ldots, X_n$ be n random variables with joint probability density function (pdf) $f(x_1, \ldots, x_n)$ and $Y = g(X_1, \ldots, X_n)$. The expected value $E(Y)$ can be obtained in one of two ways. We could directly calculate

$$Eg(X_1, \ldots, X_n) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} g(x_1, \ldots, x_n) f(x_1, \ldots, x_n) dx_1 \cdots dx_n$$