ITERATIONS ON CONVEX QUADRILATERALS

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1. Introduction. The object of this paper is to study the effect of the repeated applications of a particular process $\mathcal{P}$, when it is performed on an arbitrary (convex) quadrilateral. The process is described below.

Process $\mathcal{P}$. Given a quadrilateral $ABCD$, we construct squares on the sides AB, BC, CD, and DA [Fig. 1]. All four squares are constructed on the outside of $ABCD$. Let P_1, Q_1, R_1, and S_1 denote the centers of the squares on the sides AB, BC, CD, and DA, respectively. By joining the centers of the squares a new quadrilateral $P_1Q_1R_1S_1$ is obtained. The process of obtaining quadrilateral $P_1Q_1R_1S_1$ from quadrilateral $ABCD$ is defined as the process $\mathcal{P}$.

We will denote $P_1Q_1R_1S_1$ by $\mathcal{P}[ABCD]$ and also by Π_1. In general $P_nQ_nR_nS_n$ and Π_n will denote the quadrilateral obtained by applying the process n times. In Proposition 1 we will prove that the quadrilateral $P_1Q_1R_1S_1$ has the following properties:

(i) $P_1R_1 = Q_1S_1$, i.e. the diagonals are equal, and

(ii) P_1R_1 is perpendicular to Q_1S_1, i.e. the diagonals are perpendicular.

We note that properties (i) and (ii) are not sufficient to make $P_1Q_1R_1S_1$ a square. For our purpose we may define a square as follows. A quadrilateral $PQRS$ is a square if it has the following three properties:

(i) $PR = QS$,

(ii) PR is perpendicular to QS,

(iii) the diagonals PR and QS bisect each other.

We have seen that just one application of process $\mathcal{P}$ transforms an arbitrary quadrilateral into one which satisfies two of the three properties for a square. One wonders what effect repeated applications of $\mathcal{P}$ would have on $ABCD$. Since Π_1 satisfies (i) and (ii), it is obvious that every quadrilateral Π_n will also satisfy (i) and (ii). Let M_n and N_n denote