SOLUTIONS

No problem is ever permanently closed. Any comments, new solutions, or new insights on old problems are always welcomed by the problem editor.

45. [1992, 88] *Proposed by Mohammad K. Azarian, University of Evansville, Evansville, Indiana.*

Let α, β, and γ be the direction angles of a vector R. Without using Lagrange multipliers, show that

$$\cos\alpha + \cos\beta + \cos\gamma - 2\cos\alpha\cos\beta\cos\gamma \leq \sqrt{2}.$$

Solution by the proposer.

If we let

$$\cos\alpha = \frac{a}{\sqrt{2}}, \quad \cos\beta = \frac{b}{\sqrt{2}}, \quad \cos\gamma = \frac{c}{\sqrt{2}},$$

then we need to show that

$$a + b + c - abc \leq 2.$$

Thus, it is enough to prove that

$$4\big(4 - (a+b+c-abc)^2\big) \geq 0.$$

But, using

$$a^2 + b^2 + c^2 = 2(\cos^2\alpha + \cos^2\beta + \cos^2\gamma) = 2,$$

we have

$$\begin{aligned} 4\big(4 - (a+b+c-abc)^2\big) &= (2-2ab)(2-2ac)(2-2bc) + 4(abc)^2 \\ &= \big(c^2 + (a-b)^2\big)\big(b^2 + (a-c)^2\big)\big(a^2 + (b-c)^2\big) + 4(abc)^2. \end{aligned}$$

This completes the proof. Note that equality holds if R is perpendicular to one of the axes and is making a 45-degree angle with each of the other two axes.