SOLUTIONS

No problem is ever permanently closed. Any comments, new solutions, or new insights on old problems are always welcomed by the problem editor.

41. [1992, 27] *Proposed by Russell Euler, Northwest Missouri State University, Maryville, Missouri.*

Define the sequence $\{L_n\}_{n=1}^{\infty}$ by $L_1 = a$, $L_2 = b$ and $L_{n+2} = L_{n+1} + L_n$ for $n \geq 1$, where a and b are arbitrary integers. If $a = 1$ and $b = 2$, then $L_i = i$ for three consecutive integers i.

(i) Are there other values of a and b with this property?
(ii) Are there values of a and b such that $L_i = i$ for four consecutive values of i?
(iii)* What happens if the 'consecutive' restriction is removed in (i) and (ii)?

Solution by N. J. Kuenzi and Kandasamy Muthuvel, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin (jointly).

(i) We will answer the more general question: Are there any other values of a and b such that three consecutive terms L_n, L_{n+1}, and L_{n+2} yield three consecutive integers i, $i+1$, and $i+2$?

Let $F_1 = 1$, $F_2 = 1$, and $F_{n+2} = F_{n+1} + F_n$ for $n \geq 1$ denote the Fibonacci sequence. Then the sequence $L_1 = a$, $L_2 = b$, and $L_{n+2} = L_{n+1} + L_n$ for $n \geq 1$ can be expressed in terms of the Fibonacci sequence by

$$L_{n+2} = aF_n + bF_{n+1} \quad \text{for} \quad n \geq 1.$$

If $L_n = i$, $L_{n+1} = i+1$, and $L_{n+2} = i+2$ then $2i + 1 = i + 2$ and $i = 1$. So $a = 1$ and $b = 2$ are the only values for a and b with the property that $L_i = i$ for three consecutive integers.

Next, suppose that

$$L_n = aF_{n-2} + bF_{n-1} = 1$$
$$L_{n+1} = aF_{n-1} + bF_n = 2.$$

Eliminating the variable b from these two equations yields

$$a(F_{n-2}F_n - F_{n-1}^2) = F_n - 2F_{n-1} = -F_{n-3}.$$

Since

$$F_{n-2}F_n - F_{n-1}^2 = (-1)^{n-1},$$
$$a = (-1)^n F_{n-3}.$$