A NOWHERE ANALYTIC C^∞ FUNCTION

Kent G. Merryfield

California State University, Long Beach

1. Introduction. Early on in a typical calculus course, just after we show that a differentiable function must necessarily be continuous, we note that the converse is not true: a continuous function need not be differentiable. Of course, we provide an example of this phenomenon, usually $f(x) = |x|$. Our students should be forgiven if they are led by this example, in fact, by their whole experience, to believe that even though a continuous function may fail to be differentiable at a few "bad" points, it should still be differentiable at "most" points. After all, it would have been easy for most mathematicians in the middle of the 19th century to hold to that belief, until they were confronted with Weierstrass's example of a *nowhere* differentiable continuous function. At a later time, in a later course, we begin the discussion of power series. We show that an analytic function (by which I mean a function given locally by a power series) must necessarily be infinitely differentiable, but we note that the converse is not true: an infinitely differentiable function need not be analytic. Once again, we back up this assertion with an example, usually

$$f(x) = e^{-1/x^2} \ .$$

Once again, our students might believe that infinitely differentiable functions must nonetheless be analytic "in most places", and, once again, this belief, however reasonable, would be false. The example of

$$f(x) = e^{-1/x^2}$$

serves exactly the same purpose in the discussion of analyticity as does the example of $f(x) = |x|$ in the discussion of differentiablity. To complete the discussion of differentiability, we need Weierstrass's nowhere differentiable continuous function, and to complete