SOLUTIONS

No problem is ever permanently closed. Any comments, new solutions, or new insights on old problems are always welcomed by the problem editor.

33. [1991, 93] *Proposed by Russell Euler, Northwest Missouri State University, Maryville, Missouri.*

If A, B, and C are the angles of a triangle, prove that

$$\cot A + \cot B + \cot C \geq \sqrt{3}\ .$$

Under what conditions will equality hold?

Solution I by Bob Prielipp, University of Wisconsin-Oshkosh, Oshkosh, Wisconsin.

It is known that

$$\cot A + \cot B + \cot C \geq \frac{(a^2+b^2+c^2)(a+b+c)\sqrt{3}}{9abc}$$

(where a is the length of the side of triangle ABC opposite vertex A, b is the length of the side of triangle ABC opposite vertex B, and c is the length of the side of triangle ABC opposite vertex C), with equality holding if and only if the triangle is equilateral. [See the Solution to Problem E1861 on pp. 724–725 of the June–July 1967 issue of *The American Mathematical Monthly.*]

By the Arithmetic Mean-Geometric Mean Inequality

$$\frac{a^2+b^2+c^2}{3} \geq \sqrt[3]{a^2b^2c^2} = (abc)^{\frac{2}{3}}$$

and

$$\frac{a+b+c}{3} \geq \sqrt[3]{abc} = (abc)^{\frac{1}{3}}$$

with equality holding if and only if $a = b = c$. Thus

$$\frac{a^2+b^2+c^2}{3} \cdot \frac{a+b+c}{3} \geq abc$$