ON THE AREA INSIDE A CIRCLE

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Every calculus book in print (that I know of) calculates

$$\lim_{x \to 0} \frac{\sin x}{x} \tag{$*$}$$

in the same way. We estimate $\sin x \leq x$ by the obvious comparison of a chord against an arc of a circle (Figure 1). The lower estimate is more interesting: we majorize the area of a certain sector by the area of a triangle that contains it (Figure 2). This leads to the estimate $\sin x/x \geq \cos x$. The Pinching Theorem then implies that $\sin x/x \to 1$.

Of course this limit plays a crucial role in the calculus because it allows us to compute the derivatives of the sine and cosine functions. A few hundred pages after the derivation of this limit and its consequences it is customary to illustrate the power of integral calculus and the Fundamental Theorem by calculating the area of a circle. We do this by integrating $\sqrt{1-x^2}$. This procedure is effected by performing a trigonometric substitution, which means that we must antidifferentiate the cosine function. But we learned how to differentiate (and antidifferentiate) the cosine function by means of the limit $(*)$. And calculating the limit $(*)$ entailed knowing the area of a sector, which in turn depends on knowing the area of a circle. Obviously this is all a bit circular.

So let us resort to the familiar method of exhaustion. Define π to be the quotient of the circumference of a circle by its diameter. We agree to measure the magnitude of an angle by the length of the chord it subtends on the unit circle. (Here length is defined in the usual fashion as the limit of lengths of piecewise linear approximations.) We inscribe in the unit circle a regular polygon with n sides (Figure 3). By breaking up the polygon into triangles (Figure 4), we find that

$$n \cdot \beta = 2\pi$$

and

$$n \cdot (2\alpha + \beta) = n \cdot \pi \ .$$