A COUNTER EXAMPLE IN GROUP THEORY

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In a first course on basic group theory, one of the standard problems is to show that if $G = \{x_1, x_2, \cdots, x_n\}$ is an abelian group and n is odd then the product $x_1 x_2 \cdots x_n = e$, where e is the identity element G. In this short note, we give a counter example to show that the above result is not true if we drop the 'abelianness' of the group. In looking for an example, we do not need to consider a group of order 3, 5, 7, 11, 13, 17, 19, because these are primes and any group of prime order is cyclic and hence abelian. Also $n = 9$ does not work, because it is the square of a prime and hence the group is abelian. Also by fairly standard arguments, one can see that group of order 15 is abelian. Therefore the first possible candidate for a counter example is a group of order 21. Apart from the cyclic group of order 21, there is a unique non-abelian group of order 21 (refer to p. 112, problem 11(b) of [1]). We show that this unique non-abelian group G of order 21 works as a counter example. As a matter of fact, we find an arrangement $x_1, x_2, \cdots, x_{20}$ of non-identity elements of G such that the product $x_1 x_2 \cdots x_{20}$ is non-identity. Let $a, b \epsilon G$ such that order of a and b be 3 and 7 respectively and e be the identity element of G. Let

$$G = \{e, a, a^2, b^i, ab^i, a^2b^i : 1 \leq i \leq 6\}.$$

Since $a^{-1} = a^2$, and $\{e, b^i, 1 \leq i \leq 6\}$ is a normal subgroup of G (by Sylow Theorem), and since $ab \neq ba$, there exists an $i, 2 \leq i \leq 6$, such that $aba^2 = b^i$.

Now we let,

$$\begin{array}{lllll}
x_1 = ab^2, & x_5 = ab^4, & x_9 = ab^6, & x_{13} = a, & x_{17} = b^3, \\
x_2 = a^2b^2, & x_6 = a^2b^4, & x_{10} = a^2b^6, & x_{14} = a^2b, & x_{18} = b^4, \\
x_3 = ab^3, & x_7 = ab^5, & x_{11} = a^2, & x_{15} = b, & x_{19} = b^5, \\
x_4 = a^2b^3, & x_8 = a^2b^5, & x_{12} = ab, & x_{16} = b^2, & x_{20} = b^6.
\end{array}$$