SOLUTIONS

No problem is ever permanently closed. Any comments, new solutions, or new insights on old problems are always welcomed by the editor.

7. *Proposed by Russell Euler, Northwest Missouri State University, Maryville, Missouri.*

Evaluate

$$L = \lim_{x\to 0}\left[\frac{\sin(\tan x) - \tan(\sin x)}{\sin^{-1}(\tan^{-1} x) - \tan^{-1}(\sin^{-1} x)}\right] .$$

Solution by Robert E. Kennedy and Curtis Cooper, Central Missouri State University, Warrensburg, Missouri.

It is well-known that as $x \to 0$

$$\begin{aligned} \sin x &= x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 + O(x^9) \ , \\ \tan x &= x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + O(x^9) \ , \\ \sin^{-1} x &= x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7 + O(x^9) \ , \\ \tan^{-1} x &= x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + O(x^9) \ . \end{aligned} \tag{1}$$

Using (1), as $x \to 0$

$$\begin{aligned} \sin(\tan x) &= \tan x - \frac{1}{6}\tan^3 x + \frac{1}{120}\tan^5 x \\ &\quad - \frac{1}{5040}\tan^7 x + O(\tan^9 x) \\ &= x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + O(x^9) \\ &\quad - \frac{1}{6}\big(x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + O(x^9)\big)^3 \\ &\quad + \frac{1}{120}\big(x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + O(x^9)\big)^5 \\ &\quad - \frac{1}{5040}\big(x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + O(x^9)\big)^7 \\ &\quad + O(x^9) \end{aligned}$$