SOLUTIONS

No problem is ever permanently closed. Any comments, new solutions, or new insights on old problems are always welcomed by the editor.

3. *Proposed by Curtis Cooper and Robert E. Kennedy, Central Missouri State University, Warrensburg, Missouri.*

The Fibonacci numbers F_n satisfy $F_0 = 0$, $F_1 = 1$, and

$$F_{n+2} = F_{n+1} + F_n \text{ for } n = 0, 1, 2, \dots \ .$$

Show that

$$\sum_{i=1}^{n} F_i^3 = \frac{1}{10} F_{3n+2} + \frac{3}{5}(-1)^{n-1} F_{n-1} + \frac{1}{2} \ .$$

Solution by Joe Flowers, Northeast Missouri State University, Kirksville, Missouri and Dale Woods, Central (Oklahoma) State University, Edmond, Oklahoma (jointly).

The proof is by induction on n and use of the well-known explicit formula $F_n = \frac{1}{\sqrt{5}}(a^n - b^n)$ where $a = \frac{1+\sqrt{5}}{2}$ and $b = \frac{1-\sqrt{5}}{2}$. Note that $a \cdot b = -1$.

For $n = 1$,

$$\frac{1}{10} F_5 + \frac{3}{5} F_0 + \frac{1}{2} = \frac{5}{10} + 0 + \frac{1}{2} = 1,$$

as required. Now assume that

$$\sum_{i=1}^{k} F_i^3 = \frac{1}{10} F_{3k+2} + \frac{3}{5}(-1)^{k-1} F_{k-1} + \frac{1}{2} \text{ for some } k \geq 1 \ .$$