EVALUATION OF A FAMILY OF IMPROPER INTEGRALS

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It is easy to see that the improper integral

$$(1) \qquad I(q) = \int_1^\infty \frac{dx}{x(x^{q-1} + \cdots + x + 1)}$$

converges if $q \geq 2$. The purpose of this paper is to evaluate (1) when q is an integer.

It has been shown in [1], page 37, that

$$(2) \qquad \psi(a) - \psi(a - b) = \frac{\Gamma(a)}{\Gamma(b)} \sum_{n=1}^{\infty} \frac{\Gamma(b + n)}{n\Gamma(a + n)}$$

for $Re(a) > Re(b) \geq 0$. Equation (2) will be of particular interest when the parameters are specialized by letting $a = 1$ and $b = \frac{1}{q}$ for q = 2, 3, 4, ... For then, (2) becomes

$$\frac{\Gamma(1)}{\Gamma\left(\frac{1}{q}\right)} \sum_{n=1}^{\infty} \frac{\Gamma(\frac{1}{q} + n)}{n\Gamma(1 + n)} = \psi(1) - \psi\left(1 - \frac{1}{q}\right) ,$$

and so

$$(3) \qquad \sum_{n=1}^{\infty} \frac{\left(\frac{1}{q}\right)_n}{n!\, n} = -\gamma - \psi\left(1 - \frac{1}{q}\right) ,$$