Wave Geometry No. 46

Spin Transformations. I.

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§ 1. Introduction.

Any 4-4 matrices γ_i which satisfy the relations

$$\gamma_{(i}\gamma_{j)}=g_{ij}I \qquad i,j=1,2,3,4 \tag{1.1}$$

for any given fundamental tensor g_{ij} in a 4-dimensional Riemannian space are given as follows[1]:

$$\gamma_i=S^{-1}h_i^r\mathring{\gamma}_r S, \tag{1.2}$$

where S is any 4-4 matrix, $\mathring{\gamma}_i$ are any 4-4 matrices satisfying $\mathring{\gamma}_{(i}\mathring{\gamma}_{j)}=\delta_{ij}I$, and h_j^i satisfy the following relations:

$$\sum_{r=1}^{4} h_i^r h_j^r=g_{ij}, \tag{1.3}$$

i. e. arbitrary γ_i are given by $H=\| h_{ij}\|$ (i indicate the rows and j the columns) and a spin matrix S. Now let us consider the space Γ_4 consisting of all γ_i $(=h_i^r\mathring{\gamma}_r)$ where $\mathring{\gamma}_r$ are fixed and h_j^i may take all the values satisfying relations (1.3). An element γ_i of Γ_4 evidently satisfies (1.1). Further, let us consider the spin transformation S of the elements γ_i's of Γ_4, such that $'\gamma_i=S^{-1}\gamma_i S$. The set of all such S we write $\mathfrak{S}$; then, clearly, $\mathfrak{S}$ makes a group. So for any γ_i and $'\gamma_i$ satisfying (1.1), there exists S such that transitive group leaving Γ_4 invariant. There now arises the problem of determining group $\mathfrak{S}$.

If any two elements γ_i and $'\gamma_i$ of Γ_4 be written as follows:

$$\gamma_i=h_i^r\mathring{\gamma}_r \quad \text{and} \quad '\gamma_i=k_i^r\mathring{\gamma}_r,$$

then, from (1.3), we have $H^*H=K^*K=G$, where $H=\| h_j^i\|$, $K=\| k_j^i\|$, and $G=\| g_{ij}\|$, and the asterisk denotes the transposed matrix. If we put $HK^{-1}\equiv A$, we have $A^*A=I$, i. e. A is an orthogonal matrix. Then we say that γ_i and $'\gamma_i$ have the same or opposite orientations, according as the orthogonal matrix A is proper or improper. Specially, if we take δ_{ij} for

(1) Pauli, Ann. d. Physik. **18** (1933).
Newman, Jour. London Math. Soc. **7** (1932), p. 93.