# Compact Metric Boolean Algebras and Vector Lattices.

By

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(Received June 30, 1941.)

The main object of this paper is to show that a metric vector lattice[1] with a unit element whose any interval is compact is a sequential space. In § 1 we show that any compact metric Boolean algebra is complete and atomic. In § 2 we concern ourselves with an analogous problem for a metric complemented modular lattice. In § 3 we does with the case[2]: An Archimedean $n$-dimensional vector lattice is isomorphic with $R^n$. In § 4 the main problem, and in § 5 its special cases, are treated.

Here I express my hearty thanks to Prof. F. Maeda for his kind guidance.

**§ 1.** Let $A$ be any metric Boolean algebra of elements $x, y, z, \ldots.$ with a sharply positive functional $m[x]$, where $m[1]=1$, $m[0]=0$, and with the metric $\delta(x, y)$ introduced by $m[x]$.

THEOREM 1. In any metric Boolean algebra $A$ the following two conditions are equivalent:

(1) $A$ is compact.

(2) $A$ is complete, continuous, and atomic with an enumerable basis.

PROOF. Assume that (1) holds good. For any increasing sequence $\{x_n\}$, there exists a partial sequence $\{x_{n'}\}$ converging to some $x$. By the inequality

$$\delta(x \vee z,\ y \vee z)+\delta(x \wedge z,\ y \wedge z) \leqq \delta(x, y)\ ,^{(3)}$$

we see that the order is preserved by the metric convergence, so that $x = Vx_n$ and $m[x_n] \rightarrow m[x]$. From this we infer that $A$ is complete and continuous.[4] If $A$ is not atomic, then there exists an element containing no atomic element, which we may assume to be 1. Consider an $\varepsilon$-net $a_i$, $i=1, 2, \ldots. m$ for any given positive number $\varepsilon$. We may assume that $a_i$ is the join of some subset of an independent system $x_n$, $n=1, 2, \ldots. p$, where $\bigvee x_n=1$. Let $y_n \leqq x_n$ be an element such that $m[y_n]=\dfrac{1}{2}m[x_n]$, the existence of which will be proved easily. Put $y=\bigvee y_n$, then $\delta(y, a_i)=\dfrac{1}{2}$.

(1) For notations and terminologies see G. Birkhoff, Lattise Theory, (1940).
(2) G. Birkhoff, loc. cit., 120.
(3) G. Birkhoff, loc. cit., 42.
(4) G. Birkhoff, loc. cit., 43–44.