Ideals in a Boolean Algebra with Transfinite Chain Condition.

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Let L be a generalized $\aleph$-Boolean algebra, and Ω be an ordinal number. A subset $S=(a_\alpha;\ \alpha<\Omega)$ of L is called an ascending or descending system according as $a_\alpha<a_\beta$ or $a_\alpha>a_\beta$ for all $\alpha<\beta<\Omega$. If, for every ascending and descending system S, the power of S is $<\aleph$, then we say that L satisfies the $\aleph$-chain condition. Let $\mathfrak{a}$ be an $\aleph$-ideal in L, and $L/\mathfrak{a}$ denote the set of all equivalence classes with respect to $\mathfrak{a}$. When $L/\mathfrak{a}$ satisfies the $\aleph$-chain condition, we say that L satisfies the $\aleph$-chain condition relative to $\mathfrak{a}$, and $\mathfrak{a}$ is called a basic $\aleph$-ideal of the $\aleph$-chain condition. I shall prove that L satisfies the $\aleph$-chain condition relative to $\mathfrak{a}$ if, and only if, L satisfies the following condition:

For every $T\subset L$ such that (i) $a\in T$ implies $a\notin\mathfrak{a}$, (ii) $a, b\in T$, $a\neq b$ implies $a\frown b\in\mathfrak{a}$, the power of T is $<\aleph$.

I find also that class $\mathfrak{P}_\aleph$ of all basic $\aleph$-ideals of the $\aleph$-chain condition in L is a generalized $\aleph$-Boolean algebra, and class $\Phi^*_\aleph$ of all dual $\aleph$-ideals in $\mathfrak{P}_\aleph$ is a continuous Boolean algebra.

Next I apply this result to class $\mathfrak{F}_{\aleph_1}$ of all measure functions defined in an $\aleph_1$-Boolean algebra L. Let $\mathfrak{a}_\phi$ be the class of all a such that $\phi(a)=0$. Then $\mathfrak{a}_\phi$ is a basic $\aleph_1$-ideal of the $\aleph_1$-chain condition in L. I shall prove that class $\mathfrak{Q}_{\aleph_1}$ of all $\mathfrak{a}_\phi(\phi\in\mathfrak{F}_{\aleph_1})$ is a dual $\aleph_1$-ideal in $\mathfrak{P}_{\aleph_1}$, and therefore $\mathfrak{Q}_{\aleph_1}$ is a generalized $\aleph_1$-Boolean algebra, and class $\Psi^*_{\aleph_1}$ of all dual $\aleph_1$-ideals in $\mathfrak{Q}_{\aleph_1}$ is a continuous Boolean algebra. We shall write $\psi\prec\phi$, when $\psi(a)$ is absolutely continuous with respect to $\phi(a)$, that is, $\psi(a)=0$ for all a such that $\phi(a)=0$. Then $\psi\prec\phi$ when, and only when, $\mathfrak{a}_\psi\supset\mathfrak{a}_\phi$. Hence $\mathfrak{F}_{\aleph_1}$ is dual-isomorphic to $\mathfrak{Q}_{\aleph_1}$. Therefore $\mathfrak{F}_{\aleph_1}$ is a generalized $\aleph_1$-Boolean algebra, and the class $\Psi_{\aleph_1}$ of all $\aleph_1$-ideals in $\mathfrak{F}_{\aleph_1}$ is a continuous Boolean algebra.

Lastly I shall investigate the application to the spectral theory of the complete complex Euclidean space $\mathfrak{H}$. If a family of projections $\boldsymbol{E}(a)$ is defined for all a in an $\aleph_1$-Boolean algebra L, such that

$$(\alpha)\quad \boldsymbol{E}(a)\boldsymbol{E}(b)=\boldsymbol{0}\quad\text{when}\quad a\frown b=0,$$

$$(\beta)\quad \boldsymbol{E}(a)=\boldsymbol{E}(a_1)+\boldsymbol{E}(a_2)+\cdots\cdots+\boldsymbol{E}(a_i)+\cdots\cdots\quad\text{when}\quad a=\sum_i\oplus a_i,$$

$$(\gamma)\quad \boldsymbol{E}(1)=\boldsymbol{1};$$

then we say that $\boldsymbol{E}(a)$ is a resolution of identity in the generalized sense. Let $\mathfrak{a}_f$ be the class of all a such that $\boldsymbol{E}(a)f=0$. Then $\mathfrak{a}_f$ is a basic $\aleph_1$-ideal