On Stress-functions in General Coordinates.

By

Kakutarô MORINAGA and Takayuki NÔNO

(Received May 30, 1950)

§ 1. Stress equations of motion and Hooke's law in tensor form.

In this paper we shall consider the stress-functions in the theory of elasticity. We begin rewriting the stress equations of motion in tensor forms. We introduce a general coordinates x^i into the three dimensional Euclidean space, in which the metric is given by

$$ds^2 = g_{ij}dx^i dx^j, \tag{1.1}$$

where we sum up for $i, j=1, 2, 3$; in the following we shall use the familiar notations in tensor analysis.[1]

The equations of motion for the body in motion with acceleration f^i under body force T^i and force $T^i_{(\varphi)}$ applied over any surface $\varphi=const.$ in the body are

$$\int \rho f^i d\tau = \int \rho T^i d\tau + \int T^i_{(\varphi)} d\sigma \;^{2)} \tag{1.2}$$

and

$$\int \rho v^{[i} f^{j]} d\tau = \int \rho v^{[i} T^{j]} d\tau + \int v^{[i} T^{j]}_{(\varphi)} d\sigma, \tag{1.3}$$

where v^i is any solution of $\nabla_i v^j = \delta^j_i$ (Kronecker's delta),[3] ∇_i denotes the covariant derivative with respect to g_{ij}. Since (1.2) and (1.3) are tensor equations, and in a Cartesian coordinates coincide with the ordinary equations of motion, where v^i becomes $(x+a, y+b, z+c)$, these are the equations of motion in the general coordinates.

Let T^{ij} be the force applied over the surfaces $x^i=const.$, then by projecting the force $T^i_{(\varphi)}$ on the normal line of the surface $\varphi=const.$, we get

$$T^i_{(\varphi)} = ((\nabla_h \varphi)/\sqrt{g^{kj}\nabla_k \varphi \nabla_j \varphi}) T^{hi}, \tag{1.4}$$

1) As for tensor notations, see L. P. Eisenhart: Riemannian Geometry (1926).

2) $\int \ldots d\tau$ and $\int \ldots d\sigma$ denote the integrals with respect to the volumes and surfaces in the unstrained state respectively, and so in the following.

3) This system of equations is always integrable, since this satisfies the integrability conditions $\nabla_{[k} \delta^j_{i]} = 0$.