Representations of Orthocomplemented Modular Lattices

By

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Let L be a complemented modular lattice of order $n\geqq 4$, and $\mathfrak{S}$ the auxiliary ring of L. J.v.Neumann [1][1] has proved that L is isomorphic to the lattice $\bar{R}_{\mathfrak{S}_n}$ of all principal right ideals in $\mathfrak{S}_n$, which is the set of all n-rowed square matrices with elements in $\mathfrak{S}$. Let $a\rightarrow a^{\perp}$ be an involutory dual automorphism of L, such that $a\leqq a^{\perp}$ implies $a=o$. Then $a^{\perp}$ is the orthogonal complement of a. This dual automorphism $a\rightarrow a^{\perp}$ of L induces an involutory dual automorphism $\mathfrak{a}\rightarrow\mathfrak{a}^{\perp}$ of $\bar{R}_{\mathfrak{S}_n}$.

When L is of finite dimensions (in this case $\mathfrak{S}$ is a skew-field), Birkhoff and v.Neumann [1] has proved that there exists an involutory anti-automorphism $\alpha\rightarrow\alpha^*$ of $\mathfrak{S}$ and a definite diagonal Hermitian form $\sum_{i=1}^{n}\alpha_i^*\varphi_i\beta_i$ in $\mathfrak{S}$, such that $\varphi_i^*=\varphi_i$, aud $\sum_{i=1}^{n}\alpha_i^*\varphi_i\alpha_i=0$ implies $\alpha_i=0$ $(i=1,\cdots,n)$, and the anti-automophism $\alpha\rightarrow\alpha^*$ of $\mathfrak{S}$ generates the given automorphism $\mathfrak{a}\rightarrow\mathfrak{a}^{\perp}$ of $\bar{R}_{\mathfrak{S}_n}$. In this case we can define an inner product

$$(\mathbf{a},\ \mathbf{b})=\sum_{i=1}^{n}\alpha_i^*\varphi_i\beta_i$$

of two n-dimensional vectors $\mathbf{a}=(\alpha_1,\cdots,\alpha_n)$, $\mathbf{b}=(\beta_1,\cdots,\beta_n)$. And the orthogonality of $\mathbf{a}$ and $\mathbf{b}$ defined by $(\mathbf{a},\mathbf{b})=0$, is closely related to the orthogonal complements in L.[2]

In this paper, I shall prove that these results hold also in the general case, under the assumption that L has an *orthogonal* homogeneous basis of order $n\geqq 4$.

1. Let L be an orthocomplemented modular lattice. If we denote the orthogonal complement of a by $a^{\perp}$, then $a\rightarrow a^{\perp}$ is an involutory dual automorphism of L, such that $a\leqq a^{\perp}$ implies $a=0$.

In what follows, we shall assume that L has an orthogonal homogeneous basis of order $n\geqq 4$:

$$(a_k;k=1,\cdots,n)\perp,\qquad a_1\cup\cdots\cup a_n=1,$$
$$a_k\sim a_h\ (k,h=1,\cdots,n),\qquad a_h\leqq a_k^{\perp}\ (k\neq h).^{(3)}$$

In this case, $a_k^{\perp}=\bigvee_{h=1,h\neq k}^{n}a_h$. For, $\bigvee_{h=1,h\neq k}^{n}a_h\leqq a_k^{\perp}$, and $a_k^{\perp}$ and $\bigvee_{h=1,h\neq k}^{n}a_h$ are both complements of a_k.

2. Let $(a_k, c_{kh};k,h=1,\cdots,n)$ be a normalized frame in L, which is obtained from the orthogonal homogeneous basis $(a_k;k=1,\cdots,n)$, and let $\mathfrak{S}$ be the auxiliary ring of L relative to this frame. Then L is isomorphic to $\bar{R}_{\mathfrak{S}_n}$.[4] And the involutory dual automorphism $a\rightarrow a^{\perp}$ of L induces a unique involutory anti-automorphism $A\rightarrow A^*$ of $\mathfrak{S}_n$,[5] and

(1) The numbers in square brackets refer to the list given at the end of this paper.
(2) Cf. also Birkhoff [1] 71-72.
(3) When L is a continuous geometry of order $n\geqq 4$ with orthogonal complements, there exists such an orthogonal homogeneous basis.
(4) v.Neumann [1] Theorem 14·1.
(5) v.Neumann [1] Theorem 4·3, Theorem 4·4.