On a Theorem in Lattice Theory

By

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It is not known in the lattice theory, whether or not in general, existence and unicity of complementation imply distributivity. G. Birkhoff[1] answered positively to this question in complete, atomic lattice. But we shall show that this is true without the assumption of completeness.

THEOREM. *An atomic lattice L, each element of which has a unique complement, is a Boolean lattice.*

We denote elements of L by a, b, and atomic ones by x, y. To prove this theorem we need the following lemmas:

LEMMA 1. $a<b$ *implies* $a'\wedge b\neq 0$.

For, if $a'\wedge b=0$, then $a'\vee b\geqq a'\vee a=1$, from which we have $a=b$ by unicity of complementation. This contradicts the assumption $a<b$.

LEMMA 2. $a\nleqq b$ *implies the existence of an* x *such that* $a\wedge x=x$, $b\wedge x=0$.

For, since $a\wedge b<a$, it follows from Lemma 1 that $(a\wedge b)'\wedge a\neq 0$. So there exists an x such that $x\leqq(a\wedge b)'\wedge a$, whence $a\wedge x=x$, $b\wedge x=x\wedge(a\wedge b)'\wedge a\wedge b=0$.

LEMMA 3. $x'<a$ *implies* $a=1$.

For, using Lemma 1, $x\wedge a=(x')'\wedge a\neq 0$, whence $x\leqq a$. Thus we have $a\geqq x\vee x'=1$.

LEMMA 4. $x\vee a=0$ *implies* $x'\geqq a$.

For, $x'\ngeqq a$ implies, by Lemma 2, the existence of y such that $y\leqq a$, $y\wedge x'=0$. Using Lemma 3, $y\vee x'=1$, therefore unicity of complementation implies $x=y$, whence $0=a\wedge x=a\wedge y=y$, which is a contradiction.

LEMMA 5. *For any* x, *it holds* $x\leqq a$ *or* $x\leqq a'$.

For, the contrary implies, by Lemma 4, that $x'\geqq a$, and $x'\geqq a'$, whence we reach the contradiction that $x'\geqq a\vee a'=1$.

LEMMA 6. *For any* x, $x\leqq a\wedge b$ *implies either* $x\leqq a$ *or* $x\leqq b$.

For, the contrary implies that $x'\geqq a$, $x'\geqq b$, whence $x'\geqq a\vee b\geqq x$. So we get $x=x\wedge x'=0$ which is a contradiction.

Now we prove the theorem. We define $A_a=\{x;\ x\leqq a\}$. Using Lemma 2, 5, and 6, (1) $A_a=A_b$ if and only if $a=b$, (2) $A_{a'}=A'_a$, and (3) $A_{a\vee b}=A_a\vee A_b$. So L is isomorphic with the Boolean lattice of subsets of A_1. Therefore L is itself a Boolean lattice.

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(1) G. Birkhoff and M. Ward, Annals of Math., **40** (1939) 609-610.