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A Theorem on the Unitary Groups over Rings

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1. It is well known(1) that the unitary group $U(n, Q)$ over the sfield Q of quaternions is isomorphic to the intersection of the unitary group $U(2n, C)$ and the symplectic group $Sp(2n, C)$ over the field C of complex numbers. We shall extend this theorem for the unitary groups over rings with unit element and with an involution.

Let R be an arbitrary ring with unit element ε, and with an involution I, that is, a one-to-one mapping $\xi \rightarrow \xi^I$ of R onto itself, distinct from the identity, such that $(\xi+\eta)^I=\xi^I+\eta^I$, $(\xi\eta)^I=\eta^I\xi^I$, and $(\xi^I)^I=\xi$. Let V_R be a right and left vector space over R; an hermitian form over V_R is a mapping $(x,y)\rightarrow f(x,y)$ of $V_R\times V_R$ into R, which for any x, is linear in y, and such that $f(y,x)=f(x,y)^I$. This implies that $f(x,y)$ is additive in x and such that $f(x\lambda, y)=\lambda^I f(x,y)$. And suppose that the form f is nondegenerate, or in other words that if $f(x,y)=0$ for all $y\in V_R$, then $x=0$. A unitary transformation u of V_R is a one-to-one linear mapping of V_R onto itself such that $f(u(x), u(y))=f(x,y)$ identically, these transformations constitute the unitary group $U(V_R, f)$. As for these definitions, we followed J. Dieudonné(2).

Let $\tilde{R}$ be the ring, which is the 2-dimensional right and left vectot space over R having e_0 and e_1 as a basis; in which the multiplication is defined by (i) the distributive law, (ii) $e_0e_0=e_0$, $e_0e_1=e_1e_0=e_1$, $e_1e_1=e_0\mu$ $(\mu\in R)$, and (iii) $e_0\alpha=\alpha e_0$, $e_1\alpha^*=\alpha e_1$ for every $\alpha\in R$, where the mapping $\alpha\rightarrow\alpha^*$ is a one-to-one mapping of R onto itself; and in which the involution J is defined by $e_0{}^J=e_0$, $e_1{}^J=e_1\nu$ $(\nu\in R)$, and $\alpha^J=\alpha^I$ for every $\alpha\in R$.

Moreover let $V_{\tilde{R}}$ be the extension of V_R for the extension $\tilde{R}$ of R, then $V_{\tilde{R}}$ is the 2-dimensional right and left vector space over V_R having e_0 and e_1 as a basis. And suppose that $e_0x=xe_0$, $e_1x=x^*e_1$ for every $x\in V_R$, where the mapping $x\rightarrow x^*$ is a one-to-one mapping of V_R onto itself.

2. We shall consider the condition for the $\tilde{R}$ stated in **1** to be a ring with an involution J.

LEMMA 1. *$\tilde{R}$ is a ring, if and only if $\mu^*=\mu$ and $\mu\xi^{**}=\xi\mu$ for every $\xi\in R$.*

PROOF. For the $\tilde{R}$ stated in **1**, the multiplication is defined by $(e_0\xi_0+e_1\xi_1)(e_0\eta_0+e_1\eta_1=e_0(\xi_0\eta_0+\mu\xi_1{}^*\eta_1)+e_1(\xi_1\eta_0+\xi_0{}^*\eta_1)$; and the axioms of ring, except for the

1) Cf. C. Chevalley, *Theory of Lie groups*, Princeton University Press, 1946, p. 22.

2) J. Diedonné, *On the structure of unitary groups*, Trans. Amer. Math. Soc. vol. 72 (1952), p. 367.