JOURNAL OF SCIENCE OF THE HIROSHIMA UNIVERSITY, SER. A, VOL. 18, NO. 3, MARCH, 1955

On Splittable Linear Lie Algebras

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(Received August 6, 1954)

Introduction

Let $\mathfrak{gl}(V)$ be the Lie algebra of all linear endomorphisms of a finite-dimensional vector space V over a field K of characteristic 0. An element X of $\mathfrak{gl}(V)$ is uniqely expressed as $X=S+N$ in such a way that S is a semi-simple matrix, N is a nilpotent matrix and $[S, N]=0$. These S, N are called the semi-simple and nilpotent components of X respectively [5]. A Lie subalgebra $\mathfrak{g}$ of $\mathfrak{gl}(V)$ is called *splittable* [14] provided the components of every element of $\mathfrak{g}$ also belong to $\mathfrak{g}$. E. g., completely reducible linear Lie algebras are splittable [13]. The main purpose of this paper is to study the properties of splittable linear Lie algebras.

Necessary and sufficient conditions for $\mathfrak{g}$ to be splittable are given (Proposition 1). $\mathfrak{g}$ is called algebraic [8, p. 171] if it is the Lie algebra of an algebraic group. $Y \in \mathfrak{gl}(V)$ is called a replica of $X \in \mathfrak{gl}(V)$ [8, p. 180] if Y is contained in the Lie algebra of the smallest algebraic group whose Lie algebra contains X. Then $\mathfrak{g}$ is algebraic if and only if every replica of any element of $\mathfrak{g}$ also belongs to $\mathfrak{g}$ [8, p. 181]. Since the components of any element $X \in \mathfrak{gl}(V)$ are replicas of X. [8, p. 181], every algebraic Lie algebra is splittable. It is known [7] that if $\mathfrak{g}$ is algebraic, $\mathfrak{h}$ is its radical and $\mathfrak{n}$ is the ideal of all nilpotent matrices $\in \mathfrak{h}$, then for any Levi decomposition $\mathfrak{g}=\mathfrak{s}+\mathfrak{h}$ of $\mathfrak{g}$ there exists an abelian subalgebra $\mathfrak{a}$ of semi-simple matrices $\in \mathfrak{h}$ such that $\mathfrak{h}=\mathfrak{n}+\mathfrak{a}$, $\mathfrak{n} \cap \mathfrak{a}=0$, $[\mathfrak{s}, \mathfrak{a}]=0$. An analogue to this for splittable Lie algebras will be proved (Theorem 1). The smallest algebraic Lie algebra containing $\mathfrak{g}$ is called the algebraic hull $\mathfrak{g}^*$ of $\mathfrak{g}$ [7]. In like manner we define the splittable hull $^*\mathfrak{g}$ of $\mathfrak{g}$. It is the smallest splittable Lie algebra containing $\mathfrak{g}$. We shall show that $^*\mathfrak{g}$ is the smallest Lie algebra containing $\mathfrak{g}$ which has the same nilpotent matrices as $\mathfrak{g}^*$ (Theorem 2), and that if $\mathfrak{k}$ is a Cartan subalgebra of $\mathfrak{g}$ then $^*\mathfrak{k}$ ($\mathfrak{k}^*$) is a Cartan subalgebra of $^*\mathfrak{g}$ ($\mathfrak{g}^*$) (Proposition 5). Owing to a Cartan decomposition of $\mathfrak{g}$, M. Gotô [10] showed that $\mathfrak{g}^*$ is the direct sum of $\mathfrak{g}$ and an abelian subalgebra composed of semi-simple matrices. Without making use of a Cartan decomposition a simple proof of this result and its analogue for splittable case will be established