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*Weakly Completely Continuous Banach *-Algebras*

By

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Throughout this paper we shall be only concerned with Banach $*$-algebras with complex scalars. A topological ring is said to be dual [**4**] provided that for every closed right (left) ideal I we have $R(L(I))=I$ and $L(R(I))=I$ respectively, where L and R denote the left and right annihilators. I. Kaplansky [**7**] has shown that the following statements are equivalent for a B^*-algebra A: (1) A is dual. (2) A is a $B^*(\infty)$-sum of C^*-algebras each of which is the algebra of all completely continuous operators on a Hilbert space. (3) A has a faithful $*$-representation by completely continuous operators on a Hilbert space. (4) The socle of A is dense in A. In an earlier paper [**12**] one of the present authors proved that a completely continuous operator on a Hilbert space $\mathfrak{H}$ is characterized as a w. c. c. ($=$ weakly completely continuous) element of the algebra of operators on $\mathfrak{H}$. This leads us to show that (1)—(4) are equivalent to that (5) A is w. c. c. (§**3**).

Kaplansky [**5**] also studied the structure of c. c. B^*-algebras and obtained the result: A c. c. B^*-algebra is a $B^*(\infty)$-sum of full matrix algebras of finite orders over the complex field. This will also follow from our above-mentioned result since the algebra of completely continuous operators on a Hilbert space is finite-dimensional if and only if it is c. c. [**12**]. Various group algebras of a compact group studied by Kaplansky [**4**] are dual A^*-algebras. We show (§**4**) that every semi-simple c. c. Banach $*$-algebra in which $x^*x=0$ implies $x=0$ is an A^*-algebra considered as a dense subalgebra of a c. c. B^*-algebra. The fundamental theorem [**9**] of almost periodic functions in a group is to say that the algebra of a. p. f. is a c. c. dual A^*-algebra. In any c. c. dual A^*-algerbra every closed right ideal is the closure of the union of minimal right ideals contained in it (§**4**). Any dual B^*-algebra is c. c. if and only if it is strongly semi-simple, or the annihilator of the center is zero (§**4**).

In §**2** we treat the uniqueness problem of an auxiliary norm of A^*-algebras. We say that an A^*-algebra A has a unique auxiliary norm in case any two auxiliary norms $|x|$, $|x|_1$ are equivalent, that is, $|x_n|\rightarrow 0$ if and only if $|x_n|_1\rightarrow 0$. Whether or not every A^*-algebra has a unique auxiliary norm is open for us. We show under certain conditions that an A^*-algebra has a unique auxiliary norm.