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Extension of ♮-Application to Unbounded Operators

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In the previous paper [13], the present authors developed the so-called "non-commutative theory" of integration for rings of operators from a point of view resumed as follows. Every semi-finite ring of operators $\mathbb{M}$ with a normal, faithful and essential pseudo-trace m is normally $*$-isomorphic to the left ring $\mathbb{L}$ of an H-system $\boldsymbol{H}$ such that m corresponds to the canonical pseudo-trace of $\boldsymbol{H}$ [13]. We have shown that this $*$-isomorphism can be uniquely extended to a $*$-isomorphic mapping between the sets of measurable operators with respect to $\mathbb{M}$ and $\mathbb{L}$ respectively. Thus the theory of integration for $\mathbb{M}$ can be reduced to that for $\mathbb{L}$. But in $\boldsymbol{H}$ the set of all square-integrable measurable operators is given *a priori*, basing on which our whole theory was built.

In his investigation on ♮-applications in a ring of operators, Dixmier has shown ([4], Theorem 3) that every normal, faithful and essential pseudo-trace defined on a semi-finite ring $\mathbf{M}$ has the form $m(A)=\varphi(A^{\natural})$, where $\natural$ is a fixed normal, faithful and essential pseudo-♮-application defined on $\mathbf{M}^{+}$ and φ is a normal, faithful and essential pseudo-measure on the spectre Ω of the center $\mathbf{M}^{\natural}$. This leads us to another formulation of the theory which is divided into two parts: the classical theory of pseudo-measure on the spectre Ω of $\mathbf{M}^{\natural}$ and the extension of ♮-application to unbounded operators $\eta\mathbf{M}$. The main purpose of this paper is to develop this theory of extension. The pseudo-♮-application defined on $\mathbf{M}^{+}$, $\mathbf{M}^{+}\ni A\rightarrow A^{\natural}\in \boldsymbol{Z}$, will be extended over the set of all positive, closed, densely defined operators $T\eta\mathbf{M}$, $T\rightarrow T^{\natural}\in\boldsymbol{Z}$,

$$(\natural)\qquad T^{\natural}=\underset{\mathbf{M}^{+}\ni A\leq T}{\text{l. u. b.}}\ A^{\natural}.$$

If we wish the integral of T to be finite, $T^{\natural}$ must be finite except on a nowhere dense subset of Ω. Such a T will be measurable in the sense of Segal ([15], [13]) and the set of all such T forms the positive part of an invariant linear system $\mathfrak{S}$, which will play a fundamental rôle in our present theory.

§1 is devoted to the proof of a theorem concerning the least upper bound of an increasing directed set $\{T_\delta\}$ of positive, closed and densely defined operators