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Exceptional Values of Meromorphic Functions in a Neighborhood of the set of Singularities

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1. Let E be a compact set in the z-plane and let Ω be its complement with respect to the extended z-plane. Then it is well known that E is of capacity zero[1] if and only if Ω is a domain and admits no Green's function on it. Suppose that E is of capacity zero. We shall consider a single-valued meromorphic function $w=f(z)$ on Ω which has an essential singularity at each point of E, that is, the cluster set of $f(z)$ at each point of E is the whole w-plane.

We shall say that a value w is exceptional for $f(z)$ at a point ζ of E if there exists a neighborhood of ζ where the function does not take this value w. It is well known that the set of all exceptional values of $f(z)$ at a point ζ of E is a K_σ-set, by which we mean the union of an enumerable number of compact sets, and is of capacity zero. Here arises the following question: *Can we replace "a K_σ-set of capacity zero" by "at most two" or "at most enumerable"?*

In this paper, we shall show that, for every K_σ-set K of capacity zero in the w-plane, we can find a function $f(z)$ which has K as the set of its exceptional values at each singularity. From this fact we can of course conclude that the above question is answered in the negative. Furthermore we shall show by an example that, even if E is of logarithmic measure zero, the set of exceptional values is not always enumerable.

2. Theorem 1. *For every K_σ-set K of capacity zero in the w-plane, there exist a compact set E of capacity zero in the z-plane and a single-valued meromorphic function $f(z)$ on its complementary domain Ω such that $f(z)$ has an essential singlarity at each point of E, and the set of exceptional values at each singularity coincides with K.*

Proof.[2] Let $\{K_n\}_{n=1,2,\dots}$ be a non-decreasing sequence of compact sets whose union is equal to K and let R_n be the complement of K_n with respect to the extended w-plane. We observe first that each R_n can be considered as a Riemann surface with null boundary. The Dirichlet integral, taken in a domain (or an open set) G, of a function g will be denoted by $D_G(g)$. If, in

1) In this paper, capacity is always logarithmic.
2) When K itself is a compact set, the proof is considerably simpler. See the remark of Theorem 3.