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Note on the Paper "On the Singularity of General Linear Groups"

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§1. **Introduction.** Let G be a connected Lie group and let $\mathfrak{G}$ be its Lie algebra, then there exists an analytic mapping, which is called the exponential mapping, of the analytic manifold $\mathfrak{G}$ into the analytic manifold G. The differential of the exponential mapping at an element A of $\mathfrak{G}$ is an endomorphism $\chi(A)$ of $\mathfrak{G}$:

$$A \to \chi(A) = (\exp(-\mathrm{ad}\, A) - I)/-\mathrm{ad}\, A,$$

where $(\exp(-\mathrm{ad}\, A)-I)/-\mathrm{ad}\, A = \sum_1^\infty (-\mathrm{ad}\, A)^{m-1}/m!$ and $\mathrm{ad}\, A \cdot W = [A, W]$ ([2][1], p. 157). The endomorphism $\chi(A)$ has an inverse, if and only if $\mathrm{ad}\, A$ has no eigen values such as $2l\pi\sqrt{-1}$ (l: non-zero integers). In the previous paper "On the Singularity of General Linear Groups" ([4]), the proof of Theorem 1, which asserts that the exponential mapping is locally homeomorphic at A, if and only if $\chi(A)$ has an inverse, is unsatisfactory for the following reason. Since $\chi(A)$ is the Jacobian at A of the exponential mapping, it is clear that if $\chi(A)$ has an inverse, then the exponential mapping is locally homeomorphic at A. For the case of the complex Lie groups, from the theorem concerning the Jacobian in several complex variables ([1], p. 179), it is deduced that if $\chi(A)$ has no inverse, then the exponential mapping is not locally homeomorphic at A. But, for the case of the real Lie groups, this argument is not available. In §2 of this note we shall complete the proof of this assertion (Theorem 1), and in §3 we shall make clear the correspondence, by the exponential mapping, between the neighborhoods A and $\exp A$ respectively.

§2. Let $\mathfrak{C}_{\mathfrak{G}}(A)$ denote the centralizer of A in $\mathfrak{G}$:

$$\mathfrak{C}_{\mathfrak{G}}(A) = \{X;\ X \in \mathfrak{G},\ \mathrm{ad}\, A \cdot X = 0\},$$

and let $\mathfrak{C}_{\mathfrak{G}}(\exp A) = \{X;\ X \in \mathfrak{G},\ \exp \mathrm{ad}\, A \cdot X = X\}$. In this section we shall prove the following theorem.

THEOREM 1. *The following conditions are mutually equivalent:*

(1) *The endomorphism $\chi(A)$ of $\mathfrak{G}$ has an inverse.*

(2) *The exponential mapping of $\mathfrak{G}$ into G is locally homeomorphic at A.*

(3) $\mathfrak{C}_{\mathfrak{G}}(A) = \mathfrak{C}_{\mathfrak{G}}(\exp A)$.

1) Numbers in brackets refer to the references at the end of the paper.