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On the Lie Triple System and its Generalization

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In a Lie triple system $\mathfrak{T}$[1] (L.t.s.) over a base field Φ,[2] let D be a vector space generated from the set of linear mappings $x \rightarrow \Sigma[a_i b_i x]$ and D' be a vector subspace generated from such set of linear mappings as $[abx]=0$ for all x in $\mathfrak{T}$, then the factor space $\mathfrak{D}(\mathfrak{T}) \equiv D/D'$ has the structure of Lie algebra of linear mappings of $\mathfrak{T}$ (inner derivations algebra of $\mathfrak{T}$).

The following theorem was first established by N. Jacobson [3] and improved under weaker assumptions than his in [6].

THEOREM. *L.t.s. $\mathfrak{T}$ can be 1-to-1 imbedded into a Lie algebra $\mathfrak{L}$ in such a way that the given composition $[abc]$ in $\mathfrak{T}$ coincides with the product $[[ab]c]$ defined in $\mathfrak{L}$ and $\mathfrak{L}=\mathfrak{T}\oplus\mathfrak{D}(\mathfrak{T})$.*

$\mathfrak{L}$ is called a standard enveloping Lie algebra of $\mathfrak{T}$.

E. Cartan proved that Lie algebra is semi-simple if and only if the determinant $|g_{ij}|$ is not zero. In §1 we shall generalize this result to L.t.s., and prove some other properties. In §2, we shall define the general Lie triple system which has the geometrical meaning, and prove that the general Lie triple system can be imbedded into a Lie algebra.

§1. Some properties of Lie triple systems.

At the standard imbedding of L.t.s. $\mathfrak{T}$, $\mathfrak{T}$ and inner derivation algebra $\mathfrak{D}(\mathfrak{T})$ satisfy the following relation

$$(1.1) \qquad [\mathfrak{T}, \mathfrak{T}]=\mathfrak{D}(\mathfrak{T}),\ [\mathfrak{T}, \mathfrak{D}(\mathfrak{T})] \subseteqq \mathfrak{T},\ [\mathfrak{D}(\mathfrak{T}), \mathfrak{D}(\mathfrak{T})] \subseteqq \mathfrak{D}(\mathfrak{T}).$$

Conversely

PROPOSITION 1.1. *Let $\mathfrak{L}$ be a Lie algebra and $\mathfrak{A}$, $\mathfrak{B}$ complementary subspaces of $\mathfrak{L}$ such that $[\mathfrak{A}, \mathfrak{A}]=\mathfrak{B}$, $[\mathfrak{A}, \mathfrak{B}] \subseteqq \mathfrak{A}$, $[\mathfrak{B}, \mathfrak{B}] \subseteqq \mathfrak{B}$. Assume that there is not a non-zero element b such that $[b, \mathfrak{A}]=(0)$. Then there is a L.t.s. $\mathfrak{T}$ such that $\mathfrak{T}$ is L.t.s. isomorphic to $\mathfrak{A}$ and $\mathfrak{D}(\mathfrak{T})$ Lie isomorphic to $\mathfrak{B}$, where $\mathfrak{D}(\mathfrak{T})$ is an inner derivation algebra of $\mathfrak{T}$.*

PROOF. Put $\mathfrak{A} \equiv \mathfrak{T}$ then $\mathfrak{T}$ is a L.t.s. since $[[\mathfrak{T}, \mathfrak{T}]\mathfrak{T}] \subseteqq \mathfrak{T}$. Any element of $\mathfrak{B}$ can be written in the form $\Sigma[a, b]$ $a, b \in \mathfrak{A}$. Then it is easy to see that the mapping $\Sigma[a, b] \rightarrow \Sigma D_{(a, b)}$ $(a, b \in \mathfrak{A})$ is the Lie isomorphism of $\mathfrak{B}$ onto $\mathfrak{D}(\mathfrak{T})$:

1) The notations and terminologies used in this paper are to be found in [4] and [6].

2) Throughout the paper we shall assume that the characteristic of the base field is 0.