JOURNAL OF SCIENCE OF THE HIROSHIMA UNIVERSITY, SER. A, VOL. 20, NO. 3, FEBRUARY, 1957

# *On the Singularity of General Linear Groups*

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(Received Sept. 25, 1956)

## §1. Introduction

Let $G$ be a Lie group, $\mathfrak{G}$ its Lie algebra, then there exists the exponential mapping from $\mathfrak{G}$ into $G: X \to \exp X$, and this mapping is locally homeomorphic at the zero element $O$ of $\mathfrak{G}$. When the exponential mapping: $X \to \exp X$ is not locally homeomorphic at $X_0 \in \mathfrak{G}$, $X_0$ is called a singular point of $\mathfrak{G}$. And a set $\{\exp tX;\ t \text{ real}\}$ is called a path through the unit element $E$ of $G$.

In this paper we shall investigate the path-structure and its singularity of $\exp \mathfrak{G}$, where $\exp \mathfrak{G}$ means the image of the exponential mapping: $\exp \mathfrak{G} = \{\exp X;\ X \in \mathfrak{G}\}$. Let $R$ and $C$ be the fields of real numbers and complex numbers respectively. In §2, we have a general consideration concerning the singularity of Lie groups, and in §§3 and 4, from our standpoint we shall consider the path-structure and its singularity of the complex general linear group $GL(n, C)$ and the real general linear group $GL(n, R)$ respectively.

## §2. The singularity of Lie groups

Let $G$ be a Lie group, $\mathfrak{G}$ its Lie algebra, and $X_i (i=1,2,\cdots,r)$ be a base of $\mathfrak{G}$. Then any element $x_0$ of $\exp \mathfrak{G}$ is expressed by $x_0 = \exp \sum x_0^i X_i$, and any element $x$ of $\mathfrak{G}$ in a sufficiently small neighborhood of $x_0$ is expressed by $x = \exp \sum v^i X_i \exp \sum x_0^i X_i$, where $|v^i| (i=1,2,\cdots,r)$ are sufficiently small. The exponential mapping: $\sum x^i X_i \to \exp \sum x^i X_i$ is locally homeomorphic at $X_0 = \sum x_0^i X_i$, if and only if there exist two neighborhoods $\mathfrak{U}$ and $\mathfrak{V}$ of $O$ in $\mathfrak{G}$ which are homeomorphic by the correspondence $\sum u^i X_i \in \mathfrak{U} \leftrightarrow \sum v^i X_i \in \mathfrak{V}$ such that

(2.1) $$\exp \sum (x_0^i + u^i) X_i = \exp \sum v^i X_i \exp \sum x_0^i X_i.$$

(2.1) is written as

(2.2) $$\exp \sum (x_0^i + u^i) x_i \exp (-\sum x_0^i X_i) = \exp \sum v^i X_i.$$

From (2.2) we have ([1], p. 156)[1)]

(2.3) $$v^i = \sum J(x_0)^i_j u^j \equiv \sum ((\exp C(x_0) - E)/C(x_0))^i_j u^j,$$

1) Numbers in brackets refer to the references at the end of the paper.