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On a Theorem in an (LF) Space

By

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A locally convex space E is distinguished ([1] p. 78) if every $\sigma(E'', E')$-bounded subset of the bidual E'' is contained in the $\sigma(E'', E')$-closure of a bounded subset of E, or if the strong dual E' is barrelled (French; espace tonnelé). J. Dieudonné and L. Schwartz raised a question ([1] p. 98): is an (F) or (LF) space always distinguished? This is negatively answered by A. Grothendieck ([3] pp. 88–89). He has shown there by an example the existence of a non distinguished closed subspace of a certain distinguished (F) space. We know that the strong dual of a distinguished (F) or reflexive (LF) space is bornological ([2] p. 342). In this paper we shall generalize this in the following way: if E is a strict inductive limit of an increasing sequence of metrisable locally convex spaces E_n and is distinguished, then the strong dual E' is bornological. In this statement if each E_n is distinguished, then E is distinguished ([3] p. 85), therefore our statement contains as a special case the theorem of *A.* Grothendieck ([3] p. 85) to the effect that the strong dual of a strict inductive limit of distinguished metrisable spaces is bornological.

1. In the sequel we mean by a locally convex space a topological linear locally convex Hausdorff space. *In a locally convex space E the following statements for an absolutely convex subset U are equivalent:*

(1) *U absorbs every bounded subset of E;*

(2) *U absorbs every bounded sequence of E;*

(3) *U absorbs every sequence converging to* 0 *of E.*

If follows therefore that in a bornological locally convex space any absolutely convex subset absorbing every sequence converging to 0 is a neighborhood of 0. This remark will be used for the proof of the following Lemma. Let E be the projective limit of the sequence of locally convex spaces E_n with respect to the mappings ϕ_n, where each ϕ_n is a continuous linear mapping of E_{n+1} into E_n. Let u_n be the projection of E into E_n. We assume that u_n is onto, and therefore so also for ϕ_n. Then we have

Lemma. *If every sequence converging to* 0 *in E_n (every bounded subset of E_n) is an image of a bounded sequence (bounded subset) of E, then E is bornological.*