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On Prime Operations in the Theory of Ideals

By

Motoyoshi SAKUMA

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Introduction. This paper contains some contributions to the theory of prime operations. As is well-known, Prüfer and Krull introduced two important *prime operations*, namely the so-called *a*- and *b-operations*, in an integrally closed integral domain, and Krull proved that these two operations coincide on finitely generated ideals. But his proof seems to be neither simple nor straightforward. And the question, whether they coincide or not on arbitrary ideals, has been left open.

The main purpose of this paper is to prove that the *a*-operation is nothing but the *b*-operation. Our proof depends entirely on the existence theorem of valuations and is very simple.

In this paper, we always denote by $\mathfrak{o}$ an integral domain, and by K its field of quotients. By an $\mathfrak{o}$-ideal, we mean a fractional ideal of $\mathfrak{o}$.

1. Axioms and examples of prime operations. In his book [2, p. 118], Krull gave a system of axioms of prime operations. But, as we shall see below, his axioms are not independent. Therefore we shall begin with modifying his axioms.

Let $\mathfrak{a}$ be an $\mathfrak{o}$-ideal. A mapping $\mathfrak{a} \to \mathfrak{a}'$ ($\mathfrak{a}'$ is also an $\mathfrak{o}$-ideal) is a prime operation if it satisfies the following conditions.

P_1. $\mathfrak{a} \subseteq \mathfrak{a}'$.
P_2. $(\mathfrak{a}')' = \mathfrak{a}'$.
P_3. $\mathfrak{a} \subseteq \mathfrak{b}$ implies $\mathfrak{a}' \subseteq \mathfrak{b}'$.
P_4. $\mathfrak{o} = \mathfrak{o}'$.
P_5. $((\alpha)\mathfrak{a})' = (\alpha)\mathfrak{a}'$ for any $\alpha \in K$.

From these axioms, we deduce the following relations.

1) $(\mathfrak{a}'+\mathfrak{b}')' = (\mathfrak{a}+\mathfrak{b})'$, 2) $(\mathfrak{a}'\mathfrak{b}')' = (\mathfrak{a}\mathfrak{b})'$, 3) $(\mathfrak{a}' \cap \mathfrak{b}')' = \mathfrak{a}' \cap \mathfrak{b}'$.

In fact, 1) and 3) are immediate consequences of P_1, P_2 and P_3. As for 2), we have $\alpha\mathfrak{b} \subseteq \mathfrak{a}\mathfrak{b}$ for any $\alpha \in \mathfrak{a}$, hence $\alpha\mathfrak{b}' = ((\alpha)\mathfrak{b})' \subseteq (\mathfrak{a}\mathfrak{b})'$ by P_3 and P_5, therefore $\mathfrak{a}\mathfrak{b}' \subseteq (\mathfrak{a}\mathfrak{b})'$. From this, in the same way, $\mathfrak{a}'\mathfrak{b}' \subseteq (\mathfrak{a}\mathfrak{b}')' \subseteq (\mathfrak{a}\mathfrak{b})'' = (\mathfrak{a}\mathfrak{b})'$. Therefore $(\mathfrak{a}'\mathfrak{b}')' \subseteq (\mathfrak{a}\mathfrak{b})'' = (\mathfrak{a}\mathfrak{b})'$ by P_1 and P_2. The converse inclusion is obvious.

Next we shall give some examples of prime operations.