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Some Remarks on Zariski Rings

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Introduction. Given a Noetherian ring A with identity and an ideal $\mathfrak{m}$ of A such that $\bigcap_{n=1}^{\infty} \mathfrak{m}^n=(0)$, we may topologize A by adopting $\{\mathfrak{m}^n;\ n=1, 2,\cdots\}$ as a fundamental system of neighbourhoods of zero. This topologized ring is usually referred to as an $\mathfrak{m}$-adic ring, and is called a Zariski ring if its ideals are all closed. An $\mathfrak{m}$-adic ring is a Zariski ring if and only if $\mathfrak{m}$ is contained in its Jacobson radical, that is to say, the intersection of all its maximal ideals. In this note, unless otherwise stated, A will denote an $\mathfrak{m}$-adic Zariski ring and $\mathfrak{a}, \mathfrak{b}, \mathfrak{p}, \mathfrak{q}$ ideals of A; $\hat{A}$ will denote the completion of A and $\hat{\mathfrak{p}}, \hat{\mathfrak{q}}$ ideals of $\hat{A}$.

Now the following properties (α), (β) and (γ) are usually derived from the property $(\mathfrak{a}:cA)\hat{A}=\mathfrak{a}\hat{A}:c\hat{A}\ (c\in A)$ ([11], p. 353, Lemma 1; [9], p. 9, Proposition 1).

(α) $(\mathfrak{a}\cap\mathfrak{b})\hat{A}=\mathfrak{a}\hat{A}\cap\mathfrak{b}\hat{A}$ ([4], p. 54, Theorem 1).

(β) $(\mathfrak{a}:\mathfrak{b})\hat{A}=\mathfrak{a}\hat{A}:\mathfrak{b}\hat{A}$.

(γ) Let $\mathfrak{q}$ be $\mathfrak{p}$-primary and $\hat{\mathfrak{p}}$ be any prime divisor of $\mathfrak{q}\hat{A}$, then $\hat{\mathfrak{p}}\cap A=\mathfrak{p}$ ([1], p. 699, Proposition 6; [9], p. 9, Corollary 2).

(β) is proved as follows: $\mathfrak{a}:\mathfrak{b}=\mathfrak{a}:(b_1,\cdots, b_n)=(\mathfrak{a}:b_1A)\cap\cdots\cap(\mathfrak{a}:b_nA)$, so $(\mathfrak{a}:\mathfrak{b})\hat{A}=(\mathfrak{a}\hat{A}:b_1\hat{A})\cap\cdots\cap(\mathfrak{a}\hat{A}:b_n\hat{A})=\mathfrak{a}\hat{A}:\mathfrak{b}\hat{A}$.

Hence, from (α) and (β), we see that the mapping $\mathfrak{a}\rightarrow\mathfrak{a}\hat{A}$ is an isomorphism with respect to all the ideal-operations ($+$, $\cdot$, $:$, $\cap$).

In § 1 we shall consider some relations between the prime divisors of $\mathfrak{a}$ and those of $\mathfrak{a}\hat{A}$. For proof, in addition to the above-mentioned properties, the following fact will be used: In a Noetherian ring with identity, a prime ideal $\mathfrak{p}$ is a prime divisor of $\mathfrak{a}$ if and only if $\mathfrak{p}=\mathfrak{a}:(p)$ for some $p\notin\mathfrak{a}$.

In § 2, as an application of the results obtained in § 1, the so-called transition theorem on lengths of primary ideals will be given for Zariski rings.

In § 3, by making use of Krull's Primidealkettensatz, some relations between maximal chains of prime ideals in A and those in $\hat{A}$ will be con-