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Lengths of Projections in Rings of Operators

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Introduction

In the theory of rings of operators, Dye [1, §4] introduced in a σ-finite, essentially finite ring the concept of the length of the identity by the cyclic dissection, and proved that such a ring is expressible as a direct sum of rings such that every summand is of the uniform length, that is, all central projections in each summand have the same length ([1], Theorem 3).

In an arbitrary ring, Ogasawara [5, p. 257] defined the lengths of σ-finite projections, in connection with the lengths of normal states. In this paper, for any (not necessarily σ-finite) projection P whose central envelope is σ-finite relative to the center, we define the length by the same way as in [5], that is, the length of P is the least cardinal of the cyclic projections by whose sum P may be represented. Our definition coincides with that of [1] in the σ-finite, essentially finite case. Generalizing the result of Theorem 3 of [1], we get the decomposition theorem for length (Theorem 2), which also includes Theorem 1 of [3] as a special case.

Next we define the l-function of a ring $\boldsymbol{M}$ which is closely related to the lengths of the central σ-finite projections, and prove that the quotient of the l-function of $\boldsymbol{M}'$ divided by that of $\boldsymbol{M}$ is a unitary invariant of $\boldsymbol{M}$, which, in the semi-finite case, coincides with the unitary invariant which was introduced by Pallu de La Barrière [6], and, in the case of type III, coincides with the inverse of the coupling operator which was introduced by Griffin [3].

§ 1. Lengths of projections

A projection P in a ring $\boldsymbol{M}$ of operators on a Hilbert space $\mathfrak{H}$ is said to be *cyclic* relative to $\boldsymbol{M}$ if there exists a vector $x \in \mathfrak{H}$ such that $[\boldsymbol{M}'x] = P\mathfrak{H}$ (see Dye [1], Notation 2.1). P is said to be *σ-finite* (=countably decomposable) relative to $\boldsymbol{M}$ if any orthogonal family of non-zero projections in $\boldsymbol{M}$ and less than P is at most countable. P is σ-finite if and only if P is a sum of countable cyclic projections. If $\boldsymbol{M}$ is commutative, then any