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# Non-existence of positive commutators

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We consider the positivity of the commutator $[H, iA]$ for self-adjoint operators $H$ and $A$ in a complex Hilbert space $\mathscr{H} \neq \{0\}$. Throughout the paper we denote by $(\cdot, \cdot)$ the scalar product on $\mathscr{H}$ and by $\|\cdot\|$ the associated norm.

In the case where $H$ and $A$ are bounded, it is well known from the proof of Putnam's theorem that $[H, iA] \geq \alpha\mathbf{1}$, i.e.,

$$([H, iA]\psi, \psi) \geq \alpha\|\psi\|^2 \qquad \text{for any} \qquad \psi \in \mathscr{H},$$

is impossible for any $\alpha > 0$ (see [1; p. 61]). Our purpose in this paper is to extend the above result to the case where $H$ and $A$ are unbounded. In this case, following Mourre [2], we define the commutator $[H, iA]$ by

$$([H, iA]\psi, \phi) = i(A\psi, H\phi) - i(H\psi, A\phi), \qquad \psi, \phi \in D(A) \cap D(H),$$

where $D(A)$ (resp. $D(H)$) denotes the domain of $A$ (resp. $H$).

We prove

THEOREM. *Let $A$ and $H$ be self-adjoint operators in $\mathscr{H}$ such that $D(H) \subset D(A)$. Then $[H, iA] \geq \alpha\mathbf{1}$ is impossible for any $\alpha > 0$.*

Before proving the theorem, we give a few remarks.

REMARK 1. It follows from the closed graph theorem that the assumption of the previous result is precisely $D(H) = D(A) = \mathscr{H}$.

REMARK 2. If $D(H)$ and $D(A)$ have no inclusion relations, the conclusion in the theorem fails. For example, if $\mathscr{H} = L^2(\boldsymbol{R})$, $A = x\cdot$ with domain $D(A) = \{\psi \in \mathscr{H}; x\psi \in \mathscr{H}\}$, $H = -id/dx$ with domain $D(H) = \{\psi \in \mathscr{H}; (d/dx)\psi \in \mathscr{H}\}$, then $[H, iA] = \mathbf{1}$ on $D(A) \cap D(H)$.

PROOF OF THEOREM. In what follows $\mathscr{L}(\mathscr{H})$ denotes the Banach space of all bounded operators on $\mathscr{H}$. Suppose that $[H, iA] \geq \alpha\mathbf{1}$ holds for some $\alpha > 0$. We choose $\phi_0 \in \mathscr{H}\backslash\{0\}$ and set $\phi = (H + i)^{-1}\phi_0$. Then $\phi \in D(H)\backslash\{0\}$ and the map $\boldsymbol{R} \ni t \mapsto e^{-itH}\phi \in \mathscr{H}$ is continuously differentiable. By the closed theorem, there is a constant $C > 0$ such that

$$(1) \qquad \|A\psi\| \leq C(\|H\psi\| + \|\psi\|), \qquad \psi \in D(H).$$