HIROSHIMA MATH. J.
20 (1990), 47–55

On the derivations of generalized Witt algebras over a field of characteristic zero

Dedicated to the memory of Professor Shigeaki Tôgô

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(Received January 13, 1989)

1. Introduction

In this paper we consider the derivations of a generalized Witt algebra $W(G, I)$ over a field $\mathfrak{k}$ of characteristic zero, where I is a non-empty index set, G is an additive submonoid of $\prod_{i \in I} \mathfrak{k}_i^+$, and $\mathfrak{k}_i^+$ $(i \in I)$ are copies of the additive group $\mathfrak{k}^+$. $W(G, I)$ is a Lie algebra which has a basis $\{w(a, i) | a \in G, i \in I\}$ and the multiplication

$$[w(a, i), w(b, j)] = a_j w(a + b, i) - b_i w(a + b, j),$$

where $i, j \in I$ and $a = (a_i)_{i \in I}$, $b = (b_i)_{i \in I} \in G$.

Generalized Witt algebras have been considered by many authors over fields of positive characteristic (e.g., [4], [6], [8]) and over fields of characteristic zero (e.g., [1], [5]). We shall show that any derivation of $W(G, I)$ is a sum of a locally inner derivation and a derivation of degree zero (Theorem 1). In the case of $G = \bigoplus_{i \in I} \boldsymbol{Z}_i$ the Lie algebra $W(G, I)$ has only locally inner derivations, in particular if $|I| < \infty$ then the derivations of $W(G, I)$ are inner (Theorem 2). Concerning the above results it is known that if G is a group and L is a finitely generated G-graded Lie algebra which admits a weight space decomposition $\bigoplus_{a \in G} L_a$ with finite dimensional L_a, then a derivation of L is a sum of inner derivation and a derivation of degree zero [2, p. 36].

For every $a \in G$ let W_a be the subspace of W spanned by $\{w(a, i) | i \in I\}$. We say that a derivation δ of $W(G, I)$ has degree b if $W_a \delta \subset W_{a+b}$ for any $a \in G$, and hence every W_a is invariant under a derivation of degree zero. Let L be a Lie algebra over $\mathfrak{k}$. A derivation δ of L is a locally inner derivation if for any finite subset F of L there exist a finite-dimensional subspace V of L containing F and $x \in W$ such that $\delta|_V = \operatorname{ad} x|_V$ [3]. We denote by $\operatorname{Der}(L)$, $\operatorname{Inn}(L)$, $\operatorname{Lin}(L)$ and $\operatorname{Der}(L)_0$ respectively the derivations of L, the inner derivations of L, the locally inner derivations of L and the derivations of L of degree zero.