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On Cartan Subgroups of Linear Groups

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Introduction

Let k be an algebraically closed field of arbitrary characteristic and let $GL(n, k)$ be the group of all automorphisms of an n-dimensional vector space V over k. As usual, we introduce the Zariski topology on the space of all endomorphisms of V. For a subgroup G of $GL(n, k)$, we denote by G^* the closure of G in $GL(n, k)$. Then G^* is the smallest algebraic subgroup of $GL(n, k)$ containing G. In [8], by considering the fact that, for any connected complex linear Lie group H, the derived group of a group H^* is contained in H, we introduced the notions of D^∞-subgroups and C^∞-subgroups of $GL(n, k)$ in the following way. A subgrup G of $GL(n, k)$ is called a *D^∞-group* (resp. *C^∞-group*) provided

$$D^\infty G^* \subset G \quad (\text{resp. } C^\infty G^* \subset G)$$

where $D^\infty G^*$ (resp. $C^\infty G^*$) is the intersection of all members of the series of the derived groups $D^i G^*$ (resp. the descending central series $C^i G^*$) of a group G^*.

In [7 and 8], we introduced two kinds of "splittability" into subgroups of $GL(n, k)$. It is well known that an element x of $GL(n, k)$ can be decomposed into the Jordan product, that is, x is uniquely expressed as $x=x_s x_u$ in such a way that x_s is semisimple, x_u is unipotent and $x_s x_u = x_u x_s$. A subgroup G of $GL(n, k)$ is called splittable [7] provided every element of G can be decomposed into the Jordan product in G. Then a connected D^∞-subgroup of $GL(n, k)$ is splittable if and only if one of its maximal solvable connected subgroups is splittable [8, Theorem 4.9]. A D^∞-subgroup of $GL(n, k)$ is called to have the (S)-property provided one of its maximal solvable connected subgroups, say R, satisfies the condition that $R=TR_u$ for any maximal torus (that is, any maximal connected commutative subgroup consisting of semisimple elements) T and for the invariant subgroup R_u of all unipotent elements of R (see [8, Definitions 7.1 and 7.2]). These two kinds of "splittability" are possessed by an algebraic linear group [1, (9.2) and (12.9)] and are equivalent for a connected C^∞-group [8, Theorem 11.4]. But each of them does not imply the other for a connected D^∞-group generally [9, Examples 1 and 2].

A *Cartan subgroup* of a group G is a maximal nilpotent subgroup H such that any invariant subgroup of finite index of H is of finite index in its normalizer in G [3, p. 199]. C. Chevalley [3, Chapitre VI] and A. Borel [1, Chapitre V] investigated Cartan subgroups of a connected algebraic linear group and, in [8, Sections 9 and 12], we studied more generally Cartan sub-