J. Sci. Hiroshima Univ. Ser. A-I
25 (1961), 3-62

Theory of Errors in Numerical Integration of Ordinary Differential Equations [1)]

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(Received March 3, 1961)

Preface

Concerning errors in numerical integration of ordinary differential equations, there are three problems.

The first one is the problem of stable convergence, namely the problem of determining necessary and sufficient conditions that, for sufficiently small length of divided intervals, approximate solutions can be actually obtained by numerical integration in any finite interval where the true solution exists; and moreover, as the length of divided intervals tends to zero, these approximate solutions converge to the true solution in that interval provided all round-off errors including the errors of starting values tend to zero in a suitable manner. In this paper, we say that an integration formula is *stable* if it satisfies the above conditions. To the problem of stable convergence, so far as the author knows, an almost complete answer has been given first by G. Dahlquist [3, 4][2)] for general multi-step integration formulas. Of course, before him, the problem has been studied by many scholars, for instance, by J. Todd [14], H. Rutishauser [13], and F. B. Hildebrand [8]. But, by all of these, it has been assumed that the initial differential equations, given in the canonical form, are linear in the unknown functions with constant coefficients and moreover, even for such equations, the treatment of the problem has been illustrative rather than demonstrative. Dahlquist, on the contrary, has derived necessary conditions for general differential equations that a general multi-step integration formula may be stable and, after that, he has proved that, for any multi-step integration formula satisfying the necessary conditions derived, there actually exist numerical solutions satisfying that multi-step integration formula with any prescribed accuracy and that the numerical solutions obtained actually converge to the true solution as the length of divided intervals and the sum of round-off errors in all steps tend to zero. But he has not proved that, by means of the multi-step integration formulas satisfying his necessary conditions, for sufficiently small length of divided intervals, the numerical approximate solutions can be actually constructed in

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1) Mainly sponsored by the United States Army under Contract No. DA-11-022-ORD-2059, Mathematics Research Center, United States Army, Madison, Wisconsin.

2) The numbers in brackets refer to the references listed at the end of the paper.