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On a Space H^f

Risai SHIRAISHI and Yukio HIRATA

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Let $f(x)$ be any locally summable and positive-valued function defined almost everywhere on R^n, Euclidean n-space. Let H^f be a Hilbert space obtained by completing the space $\mathscr{S}E$, the linear space of the inverse Fourier transforms of $\mathscr{D}$, with norm $\|\xi\|_f^2=\int|\hat{\xi}(x)|^2f(x)dx$, where $\hat{\xi}$ denotes the Fourier transform of $\xi\in\mathscr{S}E$. Under more special conditions on f, the space H^f has been investigated by J. Deny [1] and B. Malgrange [5] in connection with the study of the potential theory and the theory of partial differential equations respectively. In connection with this situation, we say that f is of Deny type (simply type D) if it satisfies the condition:

(D) $$f(x), \frac{1}{f(x)} \in (1+|x|^2)^m \times L^1 \text{ for an integer } m,$$

where L^1 denotes the space of the summable functions on R^n.

We also say that f is of Malgrange type (simple type M) if it satisfies the condition:

(M) $$f(x), \frac{1}{f(x)} \leqq C(1+|x|^2)^m \text{ a.e. for a constant } C \text{ and an integer } m.$$

Actually Malgrange was concerned with the continuous f of type M.

The purpose of our investigation is to characterize these types of f by means of the properties of H^f and its related spaces.

In Section 1 we show that f is of type D if and only if H^f is a normal space of distributions. If μ is a positive measure with which we define the space H^μ in the same way as before, we can show that μ must be of the form $f(x)dx$ when H^μ is a space of distributions.

In the following sections we shall only be concerned with normal H^f. Section 2 begins with the definition of the space $H_{f,\infty}=\bigcap_s H^{f,s}$ (resp. $H'_{f,\infty}=\bigcup_s H^{f,s}$) with the topology of projective limit (resp. of inductive limit). $H^{f,s}$ stands for H^{f_1}, where $f_1(x)=(1+|x|^2)^s f(x)$ and s is a real number. Then $H_{f,\infty}$ will be a reflexive space of type (F) consisting of the distinguished elements of H^f [6], and $H'_{f,\infty}$ the anti-dual of $H_{1/f,\infty}$. We show that f is of type M if and only if $H_{f,\infty}=\mathscr{D}_{L^2}$ or $H'_{f,\infty}=\mathscr{D}'_{L^2}$.

In Section 3 we show that H^f is of local type if and only if, for some integer m, $\dfrac{1}{(1+|x-y|^2)^m}\sqrt{\dfrac{f(y)}{f(x)}}$ is a kernel of a continuous linear application of L_x^2 into L_y^2. The condition is shown to be satisfied if, for a $k(x)$ such that