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On Spectral Representations of Generalized Spectral Operators

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Introduction. In the previous paper [3], the author introduced a theory of generalized spectral operators based on spectral representations instead of spectral measures. As Foias [2] first indicated, the spectral representation corresponding to a generalized spectral or scalar operator is not uniquely determined. In fact, if we take a spectral representation U and a nilpotent operator Q: $Q^{k+1}=0$, commuting with U and if we define V by

$$V(f)=U(f)+U(Df)Q+\frac{U(D^2f)Q^2}{2}+\cdots+\frac{U(D^kf)Q^k}{k!}$$

$\left(D=\frac{1}{2}\left(\frac{\partial}{\partial\xi}+i\frac{\partial}{\partial\eta}\right), f=f(\xi,\eta)\in C_c^\infty\right)$, then U and V are different C_c^∞-spectral representations corresponding to the same scalar operator.

In the present paper, we shall show that, for two commuting spectral representations U and V corresponding to the same scalar operator, $U(f)-V(f)$ is quasi-nilpotent and in many cases, there is a relation expressed in the above form. (See §3 and §6.)

On the due course of our argument, we shall see (§4) that the operators $S_U=U(\lambda)$ and $S_U^*=U(\bar{\lambda})$ ($\lambda=\xi+i\eta$ and $\bar{\lambda}=\xi-i\eta$) together determine the representation U. Thus, in connection with our result mentioned above, we see that $S_U^*-S_V^*$ is nilpotent in a certain sense when $S_U=S_V$ and S_U^* commutes with S_V^* (§5).

We are able to consider the uniquely determined canonical representation for a scalar opertor S satisfying $S=S_U=S_U^*$ (§7). Such operators can be regarded as a generalization of Hermitian operators and will be called *real* scalar operators.

§ 1. Preliminaries.

1) *The space* $C_c^m(0\leq m\leq\infty)$. In the present paper, the basic function algebra (cf. [3]) is restricted to $C_c^m(0\leq m\leq\infty)$, the space of all complex valued m-times continuously differentiable (infinitely differentiable, if $m=\infty$) functions with compact supports on the two dimensional real space R^2. When we speak of a point of R^2 as a variable of functions, we often identify it with a point in the complex number field C, which is topologically equivalent to R^2. Thus, $f(\lambda)$ and $f(\xi,\eta)$ express the same function, where $\lambda=\xi+i\eta\in C$ and $(\xi,\eta)\in R^2$. Throughout this paper, δ always denotes a compact set and σ an