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On Affinely Connected Spaces without Conjugate Points

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Introduction. Manifolds without conjugate points have been treated in Riemannian case by M. Morse, G. A. Hedlund, E. Hopf, L. W. Green and others, and discussed in more general metric spaces by H. Busemann and E. M. Zaustinsky.

In the present paper, we shall at first extend the notion of conjugate points to differentiable manifolds with affine connection (Section 1), and investigate a two-dimensional affinely connected manifold without conjugate points. In Section 2 we shall study a condition of the non-existence of conjugate points in two-dimensional affinely connected manifold (Theorem 1). In Section 3 we shall show by an example that the result of flatness of a two-dimensional Riemannian torus without conjugate points, given by Morse, Hedlund [9] and Hopf [6] no longer holds for the affinely connected torus in general (Theorem 2).

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1. Let M be a differentiable (C^∞) manifold with an affine connection ∇. Let $\gamma(t)$ $(t \in I)$ be a geodesic in M, where I is an open interval and t is an affine parameter.

Definition. A family of curves $t \to \Gamma(t, u)$ $(t \in I,\ |u| < \delta)$ in M will be called a *geodesic deviation* of $\gamma(t)$, if $\Gamma(t, u)$ is differentiable with respect to (t, u), $\Gamma(t, 0) = \gamma(t)$ and for sufficiently small $|u|$ the curve $t \to \Gamma(t, u)$ $(t \in I)$ represents a geodesic by neglecting the order of u^2.

Let J be a relatively compact open subinterval of I. The restriction of $\gamma(t)$ on J is also written $\gamma(t)$ $(t \in J)$.

Proposition. *A family of curves $\Gamma(t, u)$ $(t \in J,\ |u| < \delta)$ containing a geodesic $\gamma(t)$ $(t \in J)$ as $\Gamma(t, 0) = \gamma(t)$ is a geodesic deviation of $\gamma(t)$, if and only if a family of vectors $Y(t) = d\Gamma\left(\frac{\partial}{\partial u}\right)_{(t,0)}$ tangent to M at $\gamma(t)$ $(t \in J)$ satisfies the following equation,*

$$(1.1) \qquad (\nabla_X \nabla_X Y - R(X, Y)X - \nabla_X (T(X, Y)))_{\gamma(t)} = 0, \qquad \textit{for } t \in J,$$

where X, Y are vector fields on M satisfying $X_{\gamma(t)} = \dot{\gamma}(t)$ and $Y_{\gamma(t)} = Y(t)$ $(t \in J)$,