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Extremal Length of Level Surfaces and Orthogonal Trajectories

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Introduction

In [6] we computed the extremal length of a family of collections of parallel segments in the (x, y)-plane. The vertical lines can be regarded as the orthogonal trajectories of the harmonic function y. From this point of view we shall generalize a part of the results in [6].

The space will be an n-dimensional space $\mathscr{E}$ in the sense of Brelot-Choquet [1]. We shall use terminologies of the case $n \geqq 3$ although the results are valid in the case $n=2$ too. In this case we need some modifications. When we say that a set is measurable in this paper, we mean that it is Lebesgue measurable.

In §1 we introduce orthogonal trajectories and regular tubes for a given harmonic function. Their existence can be proved just as for Green lines and regular tubes consisting of Green arcs, and so the proof is omitted. Then we define harmonic flows and subflows as in the two-dimensional case which was treated in [3] and [5]. The notion of extremal length (with weight) is introduced in §2 and the extremal length of harmonic subflows is computed in §3. In §4 an extremal length in a more general sense is considered and Theorems 1 and 2 in [6] are generalized. The extremal length of the family of all orthogonal trajectories is calculated in §5 with the aid of a theorem on the decomposition of the domain of definition into disjoint harmonic subflows. Finally the extremal length of level surfaces is computed in §6.

§ 1. Harmonic flows

Let G be an open set in $\mathscr{E}$ and $H(P)$ be a harmonic function in G which is not constant in any component of G. A non-empty set of the form $\{P; H(P)=\text{const.}\}$ will be called a *level surface* or an *equipotential surface* of H. It consists of a countable number of $(n-1)$-dimensional analytic surfaces, of isolated points at infinity and of an $(n-2)$-dimensional relatively closed subset of G where $\operatorname{grad} H=0$; see Lemme 12 of [1]. A point with $\operatorname{grad} H=0$ will be called *critical.* Excluding the set of all points at infinity and