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Derivations of Lie Algebras

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Introduction

Let L be a Lie algebra over a field of characteristic 0 and let $D(L)$ be the Lie algebra of all derivations of L. The problems concerning the structure of $D(L)$ and its relations with the structure of L have been investigated by several authors in [5], [7], [8], [9], [11], [14], [16], [17] etc. In a recent paper [10] G. Leger has studied the structural properties of Lie algebras L such that $D(L)=I(L)$, where $I(L)$ is the set of all inner derivations of L, and proved the following results:

(1) If the center of L is not (0) and if $D(L)=I(L)$, then L is not solvable and the radical of L is nilpotent.

(2) If the center of L is not (0) and if the nilpotent radical is quasi-cyclic, then $D(L)\neq I(L)$.

Here a nilpotent Lie algebra N is called quasi-cyclic provided N has a subspace U such that $N=U+[N, N]$ with $U\cap[N, N]=(0)$ and such that N is the direct sum of the subspaces U^i where $U^1=U$ and $U^i=[U, U^{i-1}]$ for $i\geq 2$.

We denote by $C(L)$ the set of all central derivations of L, that is, the set of all derivations of L mapping L into the center. It is the purpose of this paper to investigate the properties of Lie algebras L such that $C(L)\subset I(L)$, Lie algebras L such that $I(L)\subset C(L)$ and Lie algebras L such that $D(L)=I(L)+C(L)$, and to generalize Leger's results above.

There actually exist the Lie algebras satisfying each of these three conditions as shown in Remarks 1, 2 and 3.

In Section 2 we shall give the forms of the derivations which are at the same time inner and central. In Section 3 we shall study the Lie algebras whose central derivations are all inner. We shall show that if the center Z of L is not (0) and if $C(L)\subset I(L)^*$, the algebraic hull of $I(L)$, then for the radical R of L $\mathrm{ad}_L R$ contains no non-zero semisimple elements (Theorem 1), and that if $Z\neq(0)$ and if $C(L)\subset I(L)$ and $I(L)$ is splittable, then the radical R is nilpotent (Theorem 2). The essential part of (1) above is to assert the nilpotency of the radical and we shall show that this is a special case of our results above (Corollary to Theorem 2 and Remark 1).

In Section 4 we shall show that, when $Z\neq(0)$, $I(L)=C(L)$ if and only if