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Normal Derivatives on an Ideal Boundary

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Introduction. Let Ω be a Green space introduced by Brelot-Choquet [2]. In general, there is no notion of "smooth" boundary of Ω and we cannot define normal derivatives on "the boundary" in the usual way. Still, there are notions of Laplacian and Dirichlet integrals on Ω. Therefore, we may define "generalized" normal derivatives so as to make Green's formula valid. To be more precise, let Ω^* be a compactification of Ω so that "an ideal boundary" $\Delta=\Omega^*-\Omega$ is realized. If u is an *HD*-function (i.e., a harmonic function with finite Dirichlet integral) and if a function φ on Δ is to be a normal derivative of u, then Green's formula

$$\int_{\check{\Omega}}(\operatorname{grad} u, \operatorname{grad} f)dv = -\int_{\Delta}\varphi f\, d\sigma$$

will be satisfied for any function f with finite Dirichlet integral on Ω which is "properly" extended over Ω^*. Here, we have two points to be cleared: i) What is the measure σ, which is the surface element in the classical case? ii) What is the "proper" extension of f?

Constantinescu-Cornea [3] defined a normal derivative on the Kuramochi boundary as a measure, which corresponds to $\varphi\cdot d\sigma$ in the above argument. If we are to define a normal derivative as a function, we must specify the measure σ. Following Doob [4], we try with the harmonic measure μ. In order to assure its existence, we shall suppose that the compactification is resolutive (§2).

As for the second point, Constantinescu-Cornea [3] defined a "quasi-continuous" extension of BLD-functions over the Kuramochi compactification. The definition requires a potential theory on the compactification and is applicable only to the Kuramochi boundary. On the other hand, Doob [4] used "fine" boundary functions of *HD*-functions, which required a theory of fine limits. It also looks impossible to generalize the theory to an arbitrary compactification.

Studying these two cases closely, however, it becomes clear that we don't need such sophisticated tools as "quasi-continuity" or "fine limits". If we consider a function f on Δ which has the Dirichlet solution H_f on Ω, then the pair (f, H_f) plays the role of a "properly" extended function. Thus, our