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Extensions of Riemannian Metrics

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1 Introduction

In the present paper, we consider certain problems of extension of a Riemannian metric on a closed submanifold to the whole. Similar problems in metrizable spaces were discussed in [2], [5].

Let N be an n-dimensional submanifold of an m-dimensional, differentiable manifold M. N is said to be a *closed submanifold* if (a) it is set-theoretically closed, and (b) the topology of N as a manifold coincides with the relative topology of N in M. Let h be a Riemannian metric on N. By a *C^s-extension* of h to M is meant a Riemannian metric g of M, of class C^s, if the restriction of g to N is h.

We shall first concern with a general case for extension of a Riemannian metric on a closed submanifold (Section 2) and then prove that if M is separable and connected and N is a connected closed submanifold of M, then there exists a C^s-extension g of h so that N is totally geodesic in a strong sense under g (Section 3).

It is known that each separable, connected differentiable manifold has a bounded (or complete) Riemannian metric [4]. We shall show that if M is connected and a Riemannian metric h of a (not necessarily connected) closed submanifold is bounded (or complete), there exists a bounded (or complete) extension of h (Section 4, 5).

2 General Case

Proposition *Let M be an m-dimensional, separable, differentiable manifold of class $C^r (r \geqq 1)$, and let N be a closed submanifold of class $C^{s+1} (0 \leqq s \leqq r-1)$ with a Riemannian metric h of class C^s. Then there exists a C^s-extension of h to M.*

Proof. The condition (b) of a closed submanifold implies that each point p of N belongs to a coordinate neighborhood U in M with a coordinate system $\{u^1, u^2, \cdots, u^m\}$ such that the set $N \cap U$ is defined by the equations $u^{n+1}=0, \cdots, u^m=0$. (In the following we shall call such a coordinate system a *canonical coordinate system* of M with respect to N.) The restriction of h to $N \cap U$ is expressed by a positive definite symmetric tensor h_{ij} $(i, j=1, 2, \cdots, n)$