J. Sci. Hiroshima Univ. Ser. A-I
29 (1965), 51-74

On the Multiplicative Products of Distributions

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(Received March 1, 1965)

The present paper is a continuation of the previous paper [12] of the same title collaborated with R. Shiraishi. We investigated there multiplication between distributions centering around the two definitions of multiplicative product of two distributions; one is due to Y. Hirata and H. Ogata [3] and the other J. Mikusiński [8]. We have shown that these two definitions are entirely equivalent. In the sequel the multiplicative product of two distributions S, $T \in \mathcal{D}'(R^N)$, if it exists, will be denoted by ST. It has been pointed out there that multiplication under consideration has the following properties:

(1) if ST exists, then $(\alpha S)T$, $S(\alpha T)$ also exist for any $\alpha \in \mathcal{E}$ and

$$(\alpha S)T = S(\alpha T) = \alpha(ST),$$

(2) if $\dfrac{\partial S}{\partial x_j}T$, $j = 1, 2, \cdots, N$, exist, then ST, $S\dfrac{\partial T}{\partial x_j}$, $j=1, 2, \cdots, N$, also exist and

$$\frac{\partial}{\partial x_j}(ST) = \frac{\partial S}{\partial x_j}T + S\frac{\partial T}{\partial x_j}.$$

With necessary modifications, our treatments will also hold for distributions defined on an open subset of R^N. The multiplication is of local character. For the case $N = 1$, the first part of (1) and (2) are postulated by H. König [4] as fundamental in his axiomatic approach to a multiplication theory for distributions. It might as well be said that these properties together with local considerations express a precise statement of Schwartz's observation that the multiplicative product of two distributions is well defined if locally one is "more regular" than the other is "irregular".

On the other hand, H. G. Tillmann [13, 14] has investigated the representation theory of distributions by the boundary distributions of locally analytic functions with certain properties. In accordance with the idea of H. J. Bremermann and L. Durand [1], he suggested another approach of defining multiplication between distributions when $N=1$. Let $\hat{S}(z)$ and $\hat{T}(z)$ be locally analytic functions corresponding to S and T respectively. Putting $\hat{S}_\varepsilon(x) = \hat{S}(x + i\varepsilon) - \hat{S}(x - i\varepsilon)$ and $\hat{T}_\varepsilon(x) = \hat{T}(x + i\varepsilon) - \hat{T}(x - i\varepsilon)$, $\varepsilon > 0$, he defined