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Note on F-operators in Locally Convex Spaces

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The theory of F-operators in Banach spaces has been developed by several authors (cf. the references in [5], [7]). According to [5], a closed, normally solvable, linear mapping with finite d-characteristic is called an F-operator. It is the purpose of this paper to generalize this notion of F-operator to locally convex spaces so that we may maintain a number of the basic results known in the case of Banach spaces. For the continuous F-operators, such an attempt has been made by H. Schaefer [9] and then by A. Deprit [4]. Our main concern here is the discussion of a general theory of F-operators: characterization of F-operators, the index theorem for a product, and so on.

§ **1.** Let E and F be locally convex Hausdorff spaces (denoted by LCS). Let u be a linear mapping with domain $\mathfrak{D}_u$ in E and rang $\mathfrak{R}_u$ in F. We denote by $\mathfrak{N}_u$ the null space of u. If u is closed, $\mathfrak{N}_u$ is a closed subspace of E. u is called *open* if $u(A)$ is an open subset of $\mathfrak{R}_u$ for each open subset A of $\mathfrak{D}_u$.

A linear mapping k of E into F is called *compact* if there is a neighbourhood U of 0 in E such that the set $k(U)$ is relatively compact.

We shall say that u is an *F-operator* when (i) $\mathfrak{N}_u$ *and* $F/\mathfrak{R}_u$ *are finite dimensional*; (ii) $\mathfrak{R}_u$ *is closed*; (iii) u *is open*. Moreover if u is continuous and $\mathfrak{D}_u=E$, we shall say that u is a continuous F-operator of E into F (According to [9], u is called a σ-homomorphism). The *index* of u is defined as $\operatorname{ind} u=\dim \mathfrak{N}_u-\operatorname{codim} \mathfrak{R}_u$.

We understand by $\tilde{\mathfrak{D}}_u$ the space $\mathfrak{D}_u$ with the weakest locally convex topology which makes the identical mapping $\mathfrak{D}_u\to\mathfrak{D}_u$ and the mapping u continuous. Then u becomes a continuous mapping of $\tilde{\mathfrak{D}}_u$ into F which we shall denote by $\tilde{u}$. As shown by F. E. Browder ([3], p. 66), $\tilde{u}$ is open if and only if u is open. Therefore u is an F-operator if and only if $\tilde{u}$ is an F-operator. With this in mind, we can show

Proposition 1 ([6], Prop. 2.1.). *Let u be a closed mapping with dense domain such that the injections $\tilde{\mathfrak{D}}_u\to E$ and $\tilde{\mathfrak{D}}_{u'}\to F'$ are compact. Then u is an F-operator.*

Proof. We have only to show that $\tilde{u}$ is an F-operator. Let v, k be the mappings of $\tilde{\mathfrak{D}}_u$ into $E\times F$ defined by $v(e)=\{e, u(e)\}$ and $k(e)=\{e, 0\}$. Then v is a monomorphism with closed range and, by assumption, k is compact.