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On the Connectedness Theorem on Schemes over Local Domains

By

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The Connectedness Theorem in algebraic geometry was first proved by Zariski, using the theory of holomorphic functions on an algebraic variety, and it was applied to show the Principle of Degeneration in [7]. Later on, Chow gave "a general Connectedness Theorem" which asserts essentially that the Connectedness Theorem on a projective scheme over a complete local domain holds true. Precisely let X be a projective scheme over $Y=\mathrm{Spec}(\mathfrak{O})$, where $\mathfrak{O}$ is a complete local domain. Then if X is connected, the fiber of X at the closed point of Y is also connected. On the other hand Grothendieck gave a generalization of this theorem to a proper prescheme over a locally noetherian prescheme Y with structure morphism f. He treated the case where the direct image $f_*(\mathcal{O}_X)$ is isomorphic to $\mathcal{O}_Y$ and applied it to the case where Y is the spectrum of a "*unibranche*" local domain (cf. (III, 4.3.) in [2]). In this paper we shall also give a generalization of the Connectedness Theorem on schemes over a complete local domain (Theorem 3). Although a complete local domain is "unibranche", our result is not merely a special case of his results but covers a little more, and moreover our method is direct and elementary compared with the elaborate one adopted in [2].

The first section is devoted to a summary of some basic results on proper schemes over a local domain. In §2 we shall show the equivalence of the following two properties (P_1) and (P_2) of a local domain $\mathfrak{O}$:

(P_1) *Let X be any integral scheme, proper and dominant over* $Y=\mathrm{Spec}(\mathfrak{O})$. *Then the fiber X_{y_0} of X at the closed point y_0 of Y is connected.*

(P_2) *Let X be any integral scheme of finite type and dominant over Y. Then X is proper over Y if the fiber X_{y_0} of Y at y_0 is non-empty and proper over* $\mathrm{Spec}(\mathfrak{O}/\mathfrak{m})$, *where $\mathfrak{m}$ is the maximal ideal of $\mathfrak{O}$.*

In other words (P_1) means that "the Connectedness Theorem" on a proper scheme over Y holds true, and (P_2) means that proper morphisms to Y are characterized by the fiber over the closed point y_0 of Y. Next we shall show that any complete local domain satisfies these two properties (P_1) and (P_2), using Chow's generalization of the Connectedness Theorem mentioned as above. From these results we shall obtain a generalization of the Connectedness Theorem to schemes over a local domain in §3. Lastly in §4 we shall generalize

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