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On the Theory of the Multiplicative Products of Distributions

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Many attempts have been made to define a multiplication for distributions. H. König [9, 10] was the first to develop a systematic treatment of the subject in an abstract way, and showed that there are actually many possible multiplication theories if one gives up some of the requirements that L. Schwartz [14] has shown impossible to be satisfied. His theory is, however, rather complicated and mainly concerns with the one dimensional case. Some writers [1], [7] also worked out the theories designed for certain physical applications, where multiplication need not be commutative and the product may contain arbitrary constants.

Y. Hirata and H. Ogata [4] introduced the definition of the multiplicative product of two distributions in order to generalize the exchange formula concerning Fourier transformation. An equivalent definition of the product was given by J. Mikusiński [13]. Among the results of [18] and [6], it has been shown that given S, $T \in \mathcal{D}'(\Omega)$, where Ω is a non-empty open subset of an N-dimensional Euclidean space R^N, the product ST exists if and only if to every $\alpha \in \mathcal{D}(\Omega)$ there is a 0-neighbourhood of R^N so that $\alpha S*\check{T}$ is equivalent to a bounded measurable function continuous at 0. Here ST is defined to be a unique distribution $W \in \mathcal{D}'(\Omega)$ such that $<W, \alpha>=(\alpha S*\check{T})(0)$. The same result has been announced by J. Jelínek [8] incidentally.

The main purpose of this paper is to generalize the multiplication just considered above so as to maintain as many reasonable properties as possible. With this in mind, we reach the requirements I through IV (see Section 1 below) for multiplication between distributions. Especially the requirement IV states that multiplicative product of distributions is invariant under diffeomorphisms. The results of [6] constitute a basis and background for the present paper. With the same notations as above, if $\alpha S*\check{T}$ has the value $(\alpha S*\check{T})(0)$ at 0 in the sense of S. Łojasiewicz [12], the distribution $W \in \mathcal{D}'(\Omega)$ defined as before will be called the multiplicative product of S and T and denoted by $S \circ T$. The multiplication thus defined will satisfy the requirements indicated above. In the case $N=1$, we can make further generalizations of the notion of the multiplication. Another purpose of this paper is to investigate these multiplications.

The presentation of the material is arranged as follows: In Section 1 we write down our requirements I-IV for multiplication. Any multiplication satisfying these requirements is called normal. Section 2 contains two lem-