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On the Integral Closure of a Domain

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0. Introduction

Throughout this paper D will denote an integral domain with $1\neq 0$ and quotient field K. An element $x\in K$ is integral with respect to D provided there exist elements $a_0, a_1, \cdots, a_{n-1}$ in D such that $x^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0=0$. The integral closure D^c of D in K consists of the elements of K which are integral with respect to D, and D is said to be integrally closed in case $D=D^c$. If $x\in K$, we say that x is "almost integral" over D provided there exists $d\in D$ such that $d\neq 0$ and $dx^n\in D$ for each positive integer n. If the set D^* of almost integral elements of K over D is equal to D, then D is called "completely integrally closed" in K.

In section 1 we determine conditions in order that D^c be a one-dimensional Prufer domain, an almost Dedekind domain, and a Dedekind domain.

In [12] Krull shows that if D is a valuation ring then D is completely integrally closed if and only if D is a (proper) maximal ring in K (i.e., a rank one valuation ring). It follows that an intersection of maximal rings in K is completely integrally closed, and Krull conjectured that the converse was also true. However, an example by Nakayama [14] shows the converse to be false. In section 2 we show that the converse is true in case D satisfies a certain finiteness condition.

In general our notation and terminology will be that of [1] and [2]. In particular, we use $\subset$ to mean "contained in or equal" and $<$ to denote proper containment. An ideal A of D is proper provided $(0)<A<D$. If J is a domain with quotient field K, then J^c will denote the integral closure of J in K and J^* will denote the "complete integral closure" of J in K (i.e., the set of "almost integral" elements of K over J).

1. The Domain D^c

A domain D is called a Prufer (almost Dedekind) domain provided the quotient ring D_P is a valuation ring (rank one discrete valuation ring) for each proper prime ideal P of D (see [3], [7] and [9]). The following theorem was first proved by Gilmer in [5], however we include it here for completeness and give a different proof.

Theorem 1: *Every domain D' such that $D\subset D'<K$ is one dimensional if*