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On Some Properties of $\mathfrak{t}(n, \Phi)$ and $\mathfrak{st}(n, \Phi)$

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1. Introduction

It is known that every finite-dimensional Lie algebra L over a field Φ of arbitrary characteristic has a faithful finite-dimensional representation. If Φ is an algebraically closed field of characteristic 0, then every solvable subalgebra of $\mathfrak{gl}(n, \Phi)$ is isomorphic to a subalgebra of the Lie algebra $\mathfrak{t}(n, \Phi)$ of all the triangular matrices. Among solvable linear Lie algebras the following three Lie algebras are most familiar to us: $\mathfrak{t}(n, \Phi)$, the Lie algebra $\mathfrak{st}(n, \Phi)$ of all the triangular matrices of trace 0, and the Lie algebra $\mathfrak{n}(n, \Phi)$ of all the triangular matrices with 0's on the diagonal. B. Kostant orally informed the author that he had determined the structure of the first cohomology group $H^1(L, L)$ of $\mathfrak{n}(n, \Phi)$ and that the method of constructing an outer derivation which has been employed in the proof of Theorem 1 in [1] gives another way of finding all the nilpotent outer derivations of $\mathfrak{n}(n, \Phi)$.

It therefore seems to be an interesting problem to ask the structure of the first cohomology groups $H^1(L, L)$ of $\mathfrak{t}(n, \Phi)$ and $\mathfrak{st}(n, \Phi)$. In this paper we are concerned with this problem and show the following two theorems.

Theorem 1. *Let L be $\mathfrak{st}(n, \Phi)$ with $n \geq 2$.*

(i) *If the characteristic of Φ is 0, or if the characteristic of Φ is $p \neq 0$ and $n \not\equiv 0 \pmod p$, then $H^1(L, L) = (0)$.*

(ii) *If the characteristic of Φ is $p \neq 0$ and $n \equiv 0 \pmod p$ and if $n \geq 5$, then $\dim H^1(L, L) = n$.*

Theorem 2. *Let Φ be a field of arbitrary characteristic and let L be $\mathfrak{t}(n, \Phi)$ with $n \geq 2$. Then $\dim H^1(L, L) = n$.*

In Theorem 1 we exclude the case where the characteristic of Φ is $p \neq 0$, $n \equiv 0 \pmod p$ and $n \leq 4$. The structure of the first cohomology group $H^1(L, L)$ of $\mathfrak{st}(n, \Phi)$ in this case will be determined in Section 5.

Throughout this paper, we shall denote by Φ a field of arbitrary characteristic unless otherwise stated, and by e_0 the identity matrix in $\mathfrak{gl}(n, \Phi)$.

2. Lemmas

Throughout Sections 2, 3, 4 and 5, we denote $\mathfrak{st}(n, \Phi)$ by L for the sake