J. Sci. Hiroshima Univ. Ser. A-I
31 (1967), 147-150

Monotonicity of the Modified Likelihood Ratio Test for a Covariance Matrix

Hisao Nagao

(Received August 19, 1967)

1. Introduction and Summary

In our previous paper [4], we have proved that the modified likelihood ratio test (=modified LR test) for the equality of a covariance matrix Σ to a given one Σ_0 in a p-variate normal distribution is unbiased. The power function of this test depends only on the characteristic roots of $\Sigma\Sigma_0^{-1}$. In this note we prove that this power function is a monotonically increasing (decreasing) function of each of the characteristic roots of $\Sigma\Sigma_0^{-1}$, when it is greater (less) than one, that is, it has the monotonicity property.

2. The monotonicity of the test

Let $p\times 1$ vectors $\boldsymbol{X}_1, \cdots, \boldsymbol{X}_N$ $(N>p)$ be a random sample from a multivariate normal distribution with unknown mean vector $\boldsymbol{\mu}$ and unknown covariance matrix Σ $(\det \Sigma \neq 0)$. We wish to test the hypothesis H: $\Sigma=\Sigma_0$ against the alternatives K: $\Sigma\neq\Sigma_0$ where $\boldsymbol{\mu}$ is unknown and Σ_0 is a given positive definite matrix (p.d. matrix). The LR critical region for this problem is given by, as in Anderson [1],

$$\omega'=\left\{S \mid S \text{ is p.d. and } |S\Sigma_0^{-1}|^{\frac{N}{2}}\operatorname{etr}\left[-\frac{1}{2}\Sigma_0^{-1}S\right]\leq c_\alpha\right\}, \tag{2.1}$$

where the symbol etr means exptr, $\boldsymbol{S}=\sum_{\alpha=1}^N(\boldsymbol{X}_\alpha-\bar{\boldsymbol{X}})(\boldsymbol{X}_\alpha-\bar{\boldsymbol{X}})'$ and $\bar{\boldsymbol{X}}=N^{-1}\sum_{\alpha=1}^N\boldsymbol{X}_\alpha$. The constant c_α is determined such that the level of this test is α. By replacing $|S\Sigma_0^{-1}|^{N/2}$ to $|S\Sigma_0^{-1}|^{(N-1)/2}$ as in our previous paper [4], we can prove the following theorem.

Theorem 1. *For testing the hypothesis H: $\Sigma=\Sigma_0$ against the alternatives K: $\Sigma\neq\Sigma_0$ for unknown mean $\boldsymbol{\mu}$, the following modified LR critical region given by*

$$\omega=\left\{S \mid S \text{ is p.d. and } |S\Sigma_0^{-1}|^{\frac{n}{2}}\operatorname{etr}\left[-\frac{1}{2}\Sigma_0^{-1}S\right]\leq c_\alpha\right\} \tag{2.2}$$

has the monotonicity property with respect to each of the p-characteristic roots of $\Sigma\Sigma_0^{-1}$, that is, $\mathrm{ch}(\Sigma\Sigma_0^{-1})=(\delta_1^2, \cdots, \delta_p^2)$, where $\boldsymbol{S}=\sum_{\alpha=1}^N(\boldsymbol{X}_\alpha-\bar{\boldsymbol{X}})(\boldsymbol{X}_\alpha-\bar{\boldsymbol{X}})'$ and $n=N-1$. More precisely, the power function increases (decreases) with respect