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On Atomistic Lattices with the Covering Property

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In the investigations of lattices in geometries, a matroid lattice is defined as an upper continuous atomistic lattice with the covering property. (See [5] and [3]. A matroid lattice is called a geometric lattice in [2].) On the other hand, there exist atomistic lattices of another type with the covering property, for instance, the lattice of all closed subspaces of a normed space or a Hilbert space. Such a lattice L has the following property:

(*) Both L and its dual are atomistic and have the covering property.

In this paper, a lattice with the property (*) is called a DAC-lattice. Since it can be proved that the dual of a matroid lattice is atomistic, the defference between a matroid lattice and a DAC-lattice is that the former is upper continuous and the latter has the dual covering property.

In the literature, the properties of matroid lattices are well investigated. The main purpose of this paper is to investigate the properties of DAC-lattices compared with those of matroid lattices.

An important common property is that the modular relation is symmetric (see §2). Other common properties appear in the arguments on the perspectivity of atoms and on irreducible decomposability (see §3 and §4).

An important difference between them is that the atoms of a DAC-lattice form a projective space but those of a non-modular matroid lattice do not. For this reason, in the theory of matroid lattices parallelism is very important but in a DAC-lattice there exists only trivial parallelism, and we have an embedding theorem of DAC-lattices into projective lattice (see §5).

In the last section of this paper, we give some examples of DAC-lattices: the lattices of closed subspaces of some vector spaces, and discuss representation theorems.

§1. DAC-lattices and matroid lattices

Definition. (i) Let a and b be elements of a lattice L. We say that b *covers* a and write $a \lessdot b$ if $a<b$ and there does not exist $c \in L$ with $a<c<b$.

(ii) Let L be a lattice with 0. An element p of L is called an *atom* (or a *point*) if $0 \lessdot p$. L is called *atomic* if every non-zero element of L contains an atom. L is called *atomistic* if every element of L is the join of some set (which may be empty) of atoms. It is easy to show that L is atomistic if and only if L is relatively atomic, that is, $a<b$ implies $a<a \cup p \leqq b$ for some atom p.