J. Sci. Hiroshima Univ. Ser. A-I
32 (1968), 167-171

Some Remarks on Higher Derivations of Finite Rank in a Field of a Positive Characteristic

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(Received February 22, 1968)

In this short note we give a generalization of an approximation theorem on iterated higher derivations given by F. K. Schmidt in a paper [2] (see Satz 14). Our generalization is done by determining all the iterated higher derivations of finite rank in any field K of a positive characteristic p. The following result on a derivation d in K will play an essential role in the proof: if we have $d^p=0$, then $d^{p-1}(\alpha)=0$ if and only if $\alpha=d(\beta)$ for some β in K*). We shall give a proof of this fact using the Jacobson-Bourbaki's theorem which asserts the existence of a $1-1$ correspondence between subfields of finite codimension in a field K and certain subrings of the ring $\mathcal{L}(K)$ of endomorphisms of the additive group $(K, +)$. Lastly we shall be concerned with conditions for a purely inseparable extension K of finite degree over a field k to be a tensor product of simple extensions over k. These conditions will be given in terms of higher derivations in K.

§**1.** Let K be a field and $\mathcal{L}(K)$ the set of additive homomorphisms of K into itself. $\mathcal{L}(K)$ is considered naturally as a vector space over K. Then a sequence $\{d_i\}_{i=0,1,\dots,m}$ of elements in $\mathcal{L}(K)$ is called *a higher derivation in K of rank m* if the following conditions are satisfied: (i) *d_0 is the identity of K*, (ii) *$d_j(ab)=\sum_{i=0}^{j} d_i(a)d_{j-i}(b)$, $j=0, 1, \cdots, m$, holds for any elements a, b in K*. A higher derivation $\{d_i\}$ is called *iterated* if it satisfies one more condition (iii) *$d_i d_j=\binom{i+j}{i}d_{i+j}$ for $i+j\leqq m$ and $d_i d_j=0$ for $i+j>m$*. Let k be the subset of the elements α in K such that $d_i(\alpha)=0$ for $i\geqq 1$. Then k is a subfield of K and we call it *the constant field* of $\{d_i\}$. In the following we treat only iterated higher derivations in a field of a positive characteristic p. In this case we can easily see that a section $\{d_i\}_{i=0,1,\dots,p^e-1}$ of $\{d_i\}$ for $p^e-1\leqq m$ is also an iterated higher derivation of rank p^e-1 in K, since we have $\binom{i+j}{i}\equiv 0 \pmod p$ for $0\leqq i$, $j\leqq p^e-1$, $i+j\geqq p^e$. The following three lemmas are known.

Lemma 1. *Let $\{d_i\}_{i=0,1,\dots,m}$ be a higher derivation in K such that $d_1\neq 0$. Then we have $d_i\neq 0$ for any i and these $m+1$ elements $d_0, \cdots, d_m$ are linearly independent over K.*

*) F. K. Schmidt proved this result in a special case where K is an algebraic function field of one variable. The method of his proof is function theoretical.